Section 14.3 **Partial derivatives.**

Definition. If f is a function of two variables, its **partial derivatives** are the functions f_x and f_y defined by

$$\boxed{f_x(x,y) = \lim_{h\to 0} \frac{f(x+h,y)-f(x,y)}{h}}$$

$$\boxed{f_y(x,y) = \lim_{h\to 0} \frac{f(x,y+h)-f(x,y)}{h}}$$

Notation for partial derivatives: If $z = f(x,y)$, we write

$$f_x(x,y) = f_x = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x} f(x,y) = \frac{\partial z}{\partial x}$$

$$f_y(x,y) = f_y = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y} f(x,y) = \frac{\partial z}{\partial y}$$

Rule for finding partial derivatives of $z = f(x,y)$:

1. To find f_x, regards y as a constant and differentiate $f(x,y)$ with respect to x.
2. To find f_y, regards x as a constant and differentiate $f(x,y)$ with respect to y.

Example 1. Find the first partial derivatives of the following functions:

(a) $f(x,y) = x^4 + x^2y^2 + y^4$

$$\frac{\partial f}{\partial x} = 4x^3 + y^2(2x) + 0$$

$$\frac{\partial f}{\partial y} = 0 + x^2(2y) + 4y^3$$

(b) $f(x,y) = x^y$

$(x^n)' = nx^{n-1}$

$$\frac{\partial f}{\partial x} = yx^{y-1}$$

$(a^y)' = a^y \ln a$

$$\frac{\partial f}{\partial y} = x^y \ln x$$

(c) $f(x,y) = e^x \tan(x-y)$

$$\frac{\partial f}{\partial y} = e^x \sec^2(x-y)\frac{\partial}{\partial y}(x-y)$$
$$= e^x \sec^2(x-y)(0-1)$$
$$= -e^x \sec^2(x-y)$$

Product Rule:

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(e^x)\tan(x-y) + e^x \frac{\partial}{\partial x}(\tan(x-y))$$
$$= e^x \tan(x-y) + e^x \sec^2(x-y)\frac{\partial}{\partial x}(x-y)$$
$$= e^x \tan(x-y) + e^x \sec^2(x-y)(1)$$

$z = z(x, y)$

Example 2. Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ if z is define implicitly as a function of x and y by the equation

$$xyz = \cos(x + y + z)$$

$$\frac{\partial}{\partial x}(xyz) = \frac{\partial}{\partial x}\left[\cos(x+y+z)\right]$$

$$\frac{\partial x}{\partial x} yz + xy\frac{\partial z}{\partial x} = -\sin(x+y+z)\left(\frac{\partial}{\partial x}(x+y+z)\right)$$

$$yz + xy\frac{\partial z}{\partial x} = -\sin(x+y+z)\left(1+\frac{\partial z}{\partial x}\right)$$

solve for $\frac{\partial z}{\partial x}$

$$xy\frac{\partial z}{\partial x} + \sin(x+y+z)\frac{\partial z}{\partial x} = -\sin(x+y+z) - yz$$

$$\frac{\frac{\partial z}{\partial x}\left(xy+\sin(x+y+z)\right)}{xy+\sin(x+y+z)} = \frac{-\sin(x+y+z)-yz}{xy+\sin(x+y+z)}$$

$$\boxed{\frac{\partial z}{\partial x} = \frac{-\sin(x+y+z)-yz}{xy+\sin(x+y+z)}}$$

$$\frac{\partial}{\partial y}(xyz) = \frac{\partial}{\partial y}\cos(x+y+z)$$

$$xz + xy\frac{\partial z}{\partial y} = -\sin(x+y+z)\left(1+\frac{\partial z}{\partial y}\right)$$

solve for $\frac{\partial z}{\partial y}$

$$\boxed{\frac{\partial z}{\partial y} = \frac{-\sin(x+y+z)-xz}{xy+\sin(x+y+z)}}$$

Functions of more that two variables. Partial derivatives can also be defined for functions of three or more variables.

If f is a function of three variables x, y, and z, ten its partial derivative with respect to x can be defined as

$$\frac{\partial f}{\partial x} = f_x(x, y, z) = \lim_{h\to 0}\frac{f(x+h, y, z) - f(x, y, z)}{h}$$

and it is found by regarding y and z as constants and differentiating $f(x, y, z)$ with respect to x.

In general, if u is function of n variables, $u = f(x_1, x_2, ..., x_n)$, its partial derivative with respect to x_i is

$$\frac{\partial u}{\partial x_i} = f_{x_i}(x_1, x_2, ..., x_n) = \lim_{h\to 0}\frac{f(x_1, ..., x_i + h, , , x_n) - f(x_1, ..., x_i, , , x_n)}{h}$$

Higher derivatives. If $z = f(x, y)$, then its **second partial derivatives** are defined as

$$(f_x)_x = f_{xx} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 z}{\partial x^2}$$

$$(f_x)_y = f_{xy} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 z}{\partial x \partial y}$$

$$(f_y)_x = f_{yx} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 z}{\partial y \partial x}$$

$$(f_y)_y = f_{yy} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2} = \frac{\partial^2 z}{\partial y^2}$$

Clairaut's Theorem Suppose f is defined on a disk D that contains the point (a, b). If the functions f_{xy} and f_{yx} are both continuous on D, then

$$\boxed{f_{xy}(a, b) = f_{yx}(a, b).}$$

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Example 3. Find all the second partial derivatives for the function $f(x, y) = (x^2 + y^2)^{3/2}$

$$\frac{\partial f}{\partial x} = \frac{3}{\cancel{2}}(x^2+y^2)^{1/2}(\cancel{2}x) \qquad \frac{\partial f}{\partial y} = \frac{3}{\cancel{2}}(x^2+y^2)^{1/2}(\cancel{2}y)$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial x}\left(3x(x^2+y^2)^{1/2}\right) = 3(x^2+y^2)^{1/2} + 3x\frac{1}{\cancel{2}}(x^2+y^2)^{-1/2}(\cancel{2}x) = \frac{6x^2+3y^2}{(x^2+y^2)^{1/2}}$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial y}\left(3y(x^2+y^2)^{1/2}\right) = 3(x^2+y^2)^{1/2} + 3y\frac{1}{\cancel{2}}(x^2+y^2)^{-1/2}(\cancel{2}y) = \frac{3x^2+6y^2}{\sqrt{x^2+y^2}}$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial y}\left(3x(x^2+y^2)^{1/2}\right) = 3x \cdot \frac{1}{\cancel{2}}(x^2+y^2)^{-1/2}(\cancel{2}y) = \frac{3xy}{\sqrt{x^2+y^2}}$$

Example 4. Determine whether the function $u = e^{-x}\cos y - e^{-y}\cos x$ is a solution of Laplace's equation $u_{xx} + u_{yy} = 0$

$$\frac{\partial u}{\partial x} = u_x = \frac{\partial}{\partial x}\left(e^{-x}\cos y - e^{-y}\cos x\right)$$

$$\frac{\partial u}{\partial x} = -e^{-x}\cos y + e^{-y}(+\sin x)$$

$$\frac{\partial^2 u}{\partial x^2} = u_{xx} = \frac{\partial}{\partial x}\left(-e^{-x}\cos y + \sin x\, e^{-y}\right)$$

$$u_{xx} = e^{-x}\cos y + \cos x\, e^{-y}$$

$$\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}\left(e^{-x}\cos y - e^{-y}\cos x\right)$$

$$= -e^{-x}\sin y + e^{-y}\cos x$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}\left(-e^{-x}\sin y + e^{-y}\cos x\right)$$

$$u_{yy} = -e^{-x}\cos y - e^{-y}\cos x$$

$$u_{xx} + u_{yy} = \underbrace{\cancel{e^{-x}\cos y} + \cancel{e^{-y}\cos x}}_{u_{xx}} + \underbrace{\left(\cancel{-e^{-x}\cos y} - \cancel{e^{-y}\cos x}\right)}_{u_{yy}} = 0$$

$$\boxed{\text{YES}}$$

Example 5. Find f_{xyz} for the function $f(x,y,z) = e^{xyz}$.

$$\frac{\partial f}{\partial x} = f_x = e^{xyz}\frac{\partial}{\partial x}(xyz) = yze^{xyz}$$

$$\frac{\partial^2 f}{\partial x \partial y} = f_{xy} = \frac{\partial}{\partial y}\left(yze^{xyz}\right) \overset{\text{Product}}{\underset{\text{rule}}{=}} \frac{\partial}{\partial y}(yz)e^{xyz} + yz\frac{\partial}{\partial y}(e^{xyz})$$

$$= ze^{xyz} + yz \cdot e^{xyz}\frac{\partial}{\partial y}(xyz)$$

$$= ze^{xyz} + yze^{xyz}(xz)$$

$$f_{xy} = ze^{xyz} + xyz^2e^{xyz}$$

$$f_{xyz} = \frac{\partial}{\partial z}\left(ze^{xyz} + xyz^2e^{xyz}\right) = \frac{\partial}{\partial z}(z)e^{xyz} + z\frac{\partial}{\partial z}(e^{xyz}) + \frac{\partial}{\partial z}(xyz^2)e^{xyz} + xyz^2\frac{\partial}{\partial z}(e^{xyz})$$

$$= e^{xyz} + ze^{xyz}\frac{\partial}{\partial z}(xyz) + 2xyz\,e^{xyz} + xyz^2e^{xyz}\frac{\partial}{\partial z}(xyz)$$

$$= e^{xyz} + ze^{xyz}(xy) + 2xyze^{xyz} + xyz^2e^{xyz}(xy)$$

$$= \boxed{e^{xyz}\left[1 + 3xyz + x^2y^2z^2\right]}$$