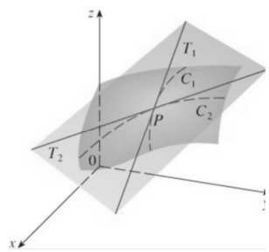
Section 14.4 **Tangent planes and linear approximations.**

Tangent planes.

- Suppose a surface S has equation $z = f(x, y)$, where f has continuous first partial derivatives, and let $P(x_0, y_0, z_0)$ be a point on S.
- Let C_1 and C_2 be the curves obtained by intersecting the vertical planes $y = y_0$ and $x = x_0$ with the surface S. P lies on both C_1 and C_2.
- Let T_1 and T_2 be the tangent lines to the curves C_1 and C_2 at the point P.
- The **tangent plane** to the surface S at the point P is defined to be the plane that containts both of the tangent lines T_1 and T_2.



An **equation on the tangent plane to the surface** $z = f(x, y)$ **at the point** $P(x_0, y_0, z_0)$ is

$$\boxed{z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)}$$

Example 1. Find the equation of the tangent plane to the surface $z = \ln(2x + y)$ at the point $(-1, 3, 0)$.

Linear Approximations.

An equation of a tangent plane to the graph of the function f of two variables at the point $(a, b, f(a, b))$ is

$$z = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

The linear function

$$L(x, y) = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

is called the **linearization** of $f(a, b)$ and the approximation

$$\boxed{f(x, y) \approx f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)}$$

is called the **linear approximation** ot the **tangent line approximation** of f at (a, b).

Example 2. Find the linear approximation for the function $f(x, y) = \sqrt{20 - x^2 - 7y^2}$ at $(2, 1)$ and use it to approximate $f(1.95, 1.08)$

Differentials. Consider a function of two variables $z = f(x, y)$. If x and y are given increments Δx and Δy, then the corresponding **increment** of z is

$$\Delta z = f(x + \Delta x, y + \Delta y) - f(x, y)$$

The increment Δz represents the change in the value of f when (x, y) changes to $(x + \Delta x, y + \Delta y)$.

The **differentials** dx and dy are independent variables. The **differential** dz (or the **total differential**), is defined by

$$\boxed{dz = f_x(x, y)dx + f_y(x, y)dy}$$

Example 3. Find the differential of the function $z = \sqrt[3]{x + y^2}$.

If we take

$$dx = \Delta x = x - a \qquad dy = \Delta y = y - b$$

then the differential of z is

$$dz = f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

On the other hand, the equation of the tangent plane to the surface $z = f(x, y)$ at the point $(a, b, f(a, b))$ is

$$z - f(a, b) = f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

We see that dz represents the change of height of the tangent plane whereas Δz represent the change in height of the surface $z = f(x, y)$ when (x, y) changes from (a, b) to $(a + \Delta x, b + \Delta y)$.

If $dx = \Delta x$ and $dy = \Delta y$ are small, then $\Delta z \approx dz$ and

$$f(a + \Delta x, b + \Delta y) \approx f(a, b) + dz$$

Functions of three or more variables. If $u = f(x, y, z)$, then the **increment** of u is

$$\Delta u = f_x(x + \Delta x, y + \Delta y, z + \Delta z) - f(x, y, z)$$

The **differential** du is defined in terms of the differentials dx, dy, and dz of the independent variables by

$$du = f_x(x, y, z)dx + f_y(x, y, z)dy + f_z(x, y, z)dz$$

If $dx = \Delta x$, $dy = \Delta y$, and $dz = \Delta z$ are small and f has continuous partial derivatives, then $\Delta u \approx du$.