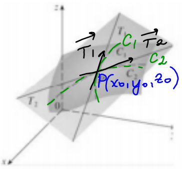
Section 14.4 **Tangent planes and linear approximations.**

Tangent planes.

- Suppose a surface S has equation $z = f(x, y)$, where f has continuous first partial derivatives, and let $P(x_0, y_0, z_0)$ be a point on S.
- Let C_1 and C_2 be the curves obtained by intersecting the vertical planes $y = y_0$ and $x = x_0$ with the surface S. P lies on both C_1 and C_2.
- Let T_1 and T_2 be the tangent lines to the curves C_1 and C_2 at the point P.
- The **tangent plane** to the surface S at the point P is defined to be the plane that containts both of the tangent lines T_1 and T_2.



An equation on the tangent plane to the surface $z = f(x, y)$ at the point $P(x_0, y_0, z_0)$ is

$$\boxed{z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)}$$

Example 1. Find ~~an~~ equation of the tangent plane to the surface $z = \ln(2x + y)$ at the point $(\overbrace{-1}^{x_0}, \underbrace{3}_{y_0}, \overbrace{0}^{z_0})$.

$$z = \ln(2x+y)$$

$$\frac{\partial z}{\partial x} = \frac{1}{2x+y}(2) \qquad \frac{\partial z}{\partial y} = \frac{1}{2x+y}$$

$$\frac{\partial z}{\partial x}(-1,3) = \frac{2}{2(-1)+3} = 2 \qquad \frac{\partial z}{\partial y}(-1,3) = \frac{1}{2(-1)+3} = 1$$

Tangent plane: $\boxed{z - 0 = 2(x+1) + 1(y-3)}$

Linear Approximations.

An equation of a tangent plane to the graph of the function f of two variables at the point $(a, b, f(a, b))$ is

$$z = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

The linear function

$$L(x, y) = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

is called the **linearization** of $f(a, b)$ and the approximation

$$\boxed{f(x, y) \approx f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)}$$

is called the linear approximation or the **tangent ~~line~~ plane approximation** of f at (a, b).

Example 2. Find the linear approximation for the function $f(x, y) = \sqrt{20 - x^2 - 7y^2}$ at $(\overbrace{2}^{a}, \underbrace{1}_{b})$ and use it to approximate $f(1.95, 1.08)$

$$f(x,y) \approx f(2,1) + f_x(2,1)(x-2) + f_y(2,1)(y-1)$$

$$f(x,y) = \sqrt{20 - x^2 - 7y^2} \qquad f(2,1) = \sqrt{20-4-7} = \sqrt{9} = 3$$

$$f_x(x,y) = \frac{\partial f}{\partial x} = \frac{1}{\cancel{2}}(20 - x^2 - 7y^2)^{-1/2}(-\cancel{2}x) \qquad f_x(2,1) = -\frac{2}{3}$$

$$f_y(x,y) = \frac{\partial f}{\partial y} = -\frac{1}{\cancel{2}}(20 - x^2 - 7y^2)^{-1/2}(-\cancel{14}^{\,7}y) \qquad f_y(2,1) = -\frac{7}{3}$$

Linear approximation: $\boxed{f(x,y) \approx 3 - \frac{2}{3}(x-2) - \frac{7}{3}(y-1)}$

$$f(1.95, 1.08) \approx 3 - \frac{2}{3}(1.95-2) - \frac{7}{3}(1.08-1) = 3 - \frac{2}{3}(-0.05) - \frac{7}{3}(0.08)$$

$$= 3 + \frac{0.1}{3} - \frac{0.56}{3} = 3 - \frac{0.46}{3} = \boxed{\frac{8.54}{3}}$$

Differentials. Consider a function of two variables $z = f(x, y)$. If x and y are given increments Δx and Δy, then the corresponding **increment** of z is

$$\Delta z = f(x + \Delta x, y + \Delta y) - f(x, y)$$

The increment Δz represents the change in the value of f when (x, y) changes to $(x + \Delta x, y + \Delta y)$.

The **differentials** dx and dy are independent variables. The **differential** dz (or the **total differential**), is defined by

$$\boxed{dz = f_x(x, y)dx + f_y(x, y)dy}$$

Example 3. Find the differential of the function $z = \sqrt[3]{x + y^2}$. $= (x+y^2)^{1/3}$

$$\frac{\partial z}{\partial x} = \frac{1}{3}(x+y^2)^{-2/3}$$

$$\frac{\partial z}{\partial y} = \frac{1}{3}(x+y^2)^{-2/3}(2y)$$

$$\boxed{dz = \frac{1}{3}(x+y^2)^{-2/3}dx + \frac{1}{3}(x+y^2)^{-2/3}(2y)\,dy}$$

If we take

$$dx = \Delta x = x - a \qquad dy = \Delta y = y - b$$

then the differential of z is

$$dz = f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

On the other hand, the equation of the tangent plane to the surface $z = f(x, y)$ at the point $(a, b, f(a, b))$ is

$$z - f(a, b) = f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

We see that dz represents the change of height of the tangent plane whereas Δz represent the change in height of the surface $z = f(x, y)$ when (x, y) changes from (a, b) to $(a + \Delta x, b + \Delta y)$.

If $dx = \Delta x$ and $dy = \Delta y$ are small, then $\Delta z \approx dz$ and

$$\boxed{f(a + \Delta x, b + \Delta y) \approx f(a, b) + dz}$$

Functions of three or more variables. If $u = f(x, y, z)$, then the **increment** of u is

$$\Delta u = f_x(x + \Delta x, y + \Delta y, z + \Delta z) - f(x, y, z)$$

The **differential** du is defined in terms of the differentials dx, dy, and dz of the independent variables by

$$\boxed{du = f_x(x, y, z)dx + f_y(x, y, z)dy + f_z(x, y, z)dz}$$

If $dx = \Delta x$, $dy = \Delta y$, and $dz = \Delta z$ are small and f has continuous partial derivatives, then $\Delta u \approx du$.