

Section 14.5 **The chain rule.**

For a function of a single variable $y = f(x)$ and $x = g(t)$, then $\frac{dy}{dt} = \frac{dy}{dx} \frac{dx}{dt}$.

The Chain Rule (case 1). Suppose that $z = f(x, y)$ is a differentiable function of x and y , where $x = g(t)$ and $y = h(t)$ are both differentiable functions of t . Then z is a differentiable function of t and

$$\boxed{\frac{dz}{dt} = \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt}}$$

Example 1. If $z = x^2y^3 + x^3y^2$, where $x = 1 + \sqrt{t}$ and $y = 1 + e^{2t}$, find $\frac{dz}{dt}$.

Example 2. The radius of a right circular cylinder is decreasing at a rate of 1.2 cm/s while its height is increasing at a rate of 3 cm/s. At what rate is the volume of the cylinder changing when the radius is 80 cm and the height is 150 cm.

The Chain Rule (case 2). Suppose that $z = f(x, y)$ is a differentiable function of x and y , where $x = g(s, t)$, $y = h(s, t)$, and the partial derivatives g_s, g_t, h_s , and h_t exist. Then

$$\boxed{\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial s}} \quad \boxed{\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial t}}$$

Example 3. If $z = x^2 \sin y$, where $x = s^2 + t^2$ and $y = 2st$, find $\frac{\partial z}{\partial s}$ and $\frac{\partial z}{\partial t}$.

The Chain Rule (general version). Suppose that u is a differentiable function of the n variables x_1, x_2, \dots, x_n and each x_j is a function of the m variables t_1, t_2, \dots, t_m such that all partial derivatives $\partial x_j / \partial t_i$ exist ($j = 1, 2, \dots, n$, $i = 1, 2, \dots, m$). Then u is a function of t_1, t_2, \dots, t_m and

$$\boxed{\frac{\partial u}{\partial t_i} = \frac{\partial u}{\partial x_1} \frac{\partial x_1}{\partial t_i} + \frac{\partial u}{\partial x_2} \frac{\partial x_2}{\partial t_i} + \dots + \frac{\partial u}{\partial x_n} \frac{\partial x_n}{\partial t_i}}$$

for each $i = 1, 2, \dots, m$.

Example 4. If $z = \frac{x}{y}$, where $x = re^{st}$, $y = rse^t$, find $\frac{\partial z}{\partial r}$, $\frac{\partial z}{\partial s}$, and $\frac{\partial z}{\partial t}$.

Implicit differentiation. We suppose that an equation of the form $F(x, y) = 0$ defines y implicitly as a differentiable function of x , that is $y = f(x)$, where $F(x, f(x)) = 0$ for all x in the domain of f .

In order to find dy/dx , we differentiate both parts of the equation $F(x, y) = 0$ with respect to x :

$$\frac{d}{dx}(F(x, y)) = \frac{d}{dx}(0)$$

$$\frac{\partial F}{\partial x} \frac{dx}{dx} + \frac{\partial F}{\partial y} \frac{dy}{dx} = 0$$

Since $\frac{dx}{dx} = 1$, then

$$\boxed{\frac{dy}{dx} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}}}$$

Example 5. Find dy/dx if $x^2 - xy + y^3 = 8$.

Suppose that z is given implicitly as a function $z = f(x, y)$ by an equation of the form $F(x, y, z) = 0$. If F is differentiable and f_x and f_y exist, then we can use the Chain Rule to differentiate the equation $F(x, y, z) = 0$ as follows

$$\frac{\partial F}{\partial x} \frac{\partial x}{\partial x} + \frac{\partial F}{\partial y} \frac{\partial y}{\partial x} + \frac{\partial F}{\partial z} \frac{\partial z}{\partial x} = 0$$

Since $\frac{\partial x}{\partial x} = 1$ and $\frac{\partial y}{\partial x} = 0$, then

$$\frac{\partial F}{\partial x} + \frac{\partial F}{\partial z} \frac{\partial z}{\partial x} = 0$$

or

$$\boxed{\frac{\partial z}{\partial x} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial z}}} \quad \boxed{\frac{\partial z}{\partial y} = -\frac{\frac{\partial F}{\partial y}}{\frac{\partial F}{\partial z}}}$$

Example 6. Find $\partial z/\partial x$ and $\partial z/\partial y$ if $xyz = \cos(x + y + z)$.