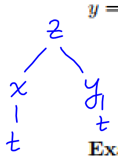
Section 14.5 **The chain rule.** $y = f(x(t))$

For a function of a single variable $y = f(x)$ and $x = g(t)$, then $\frac{dy}{dt} = \frac{df}{dx}\frac{dx}{dt}$.

$z = z(x(t), y(t))$

The Chain Rule (case 1). Suppose that $z = f(x, y)$ is a differentiable function of x and y, where $x = g(t)$ and $y = h(t)$ are both differentiable functions of t. Then z is a differentiable function of t and



$$\boxed{\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}}$$

Example 1. If $z = x^2y^3 + x^3y^2$, where $x = 1 + \sqrt{t}$ and $y = 1 + e^{2t}$, find $\frac{dz}{dt}$.

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}x'(t) + \frac{\partial z}{\partial y}y'(t)$$

$\frac{\partial z}{\partial x} = 2xy^3 + 3x^2y^2$ | $x'(t) = \frac{1}{2\sqrt{t}}$ | $y'(t) = 2e^{2t}$

$\frac{\partial z}{\partial y} = 3x^2y^2 + 2x^3y$

$$\boxed{\frac{dz}{dt} = (2xy^3 + 3x^2y^2)\frac{1}{2\sqrt{t}} + (3x^2y^2 + 2x^3y)\,2e^{2t}}$$

Example 2. The radius of a right circular cylinder is decreasing at a rate of 1.2 cm/s while its height is increasing at a rate of 3 cm/s. At what rate is the volume of the cylinder changing when the radius is 80 cm and the height is 150 cm.

$r = r(t)$, $h = h(t)$

$\frac{dr}{dt} = -1.2$, $\frac{dh}{dt} = 3$

$\frac{dV}{dt}$ when $r = 80$ and $h = 150$

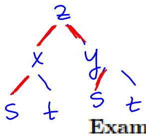
$V = 2\pi r^2 h$

$$\frac{dV}{dt} = \frac{\partial V}{\partial r}\frac{dr}{dt} + \frac{\partial V}{\partial h}\frac{dh}{dt}$$

$$= 4\pi r h \frac{dr}{dt} + 2\pi r^2 \frac{dh}{dt}$$

$$= \boxed{4\pi(80)(150)(-1.2) + 2\pi(80^2)(3)}$$

The Chain Rule (case 2). Suppose that $z = f(x, y)$ is a differentiable function of x and y, where $x = g(s,t)$, $y = h(s,t)$, and the partial derivatives g_s, g_t, h_s, and h_t exist. Then



$$\boxed{\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}} \qquad \boxed{\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}}$$

Example 3. If $z = x^2 \sin y$, where $x = s^2 + t^2$ and $y = 2st$, find $\frac{\partial z}{\partial s}$ and $\frac{\partial z}{\partial t}$.

$$\frac{\partial z}{\partial x} = 2x \sin y \qquad \frac{\partial x}{\partial s} = 2s \qquad \frac{\partial y}{\partial s} = 2t$$

$$\frac{\partial z}{\partial y} = x^2 \cos y \qquad \frac{\partial x}{\partial t} = 2t \qquad \frac{\partial y}{\partial t} = 2s$$

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} = (2x \sin y)(2s) + (x^2 \cos y)(2t)$$

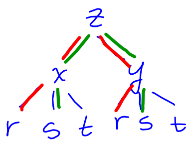
$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t} = (2x \sin y)(2t) + (x^2 \cos y)(2s)$$

The Chain Rule (general version). Suppose that u is a differentiable function of the n variables $x_1, x_2, \ldots, x_n$ and each x_j is a function of the m variables $t_1, t_2, \ldots, t_m$ such that all partial derivatives $\partial x_j / \partial t_i$ exist ($j = 1, 2, \ldots, n$, $i = 1, 2, \ldots, m$). Then u is a function of $t_1, t_2, \ldots, t_m$ and

$$\boxed{\frac{\partial u}{\partial t_i} = \frac{\partial u}{\partial x_1}\frac{\partial x_1}{\partial t_i} + \frac{\partial u}{\partial x_2}\frac{\partial x_2}{\partial t_i} + \ldots + \frac{\partial u}{\partial x_n}\frac{\partial x_n}{\partial t_i}}$$

for each $i = 1, 2, \ldots, m$.

Example 4. If $z = \frac{x}{y}$, where $x = re^{st}$, $y = rse^t$, find $\frac{\partial z}{\partial r}$, $\frac{\partial z}{\partial s}$, and $\frac{\partial z}{\partial t}$.



$$\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial r} = \frac{1}{y} e^{st} - \frac{x}{y^2} s e^t$$

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} = \frac{1}{y} rte^{st} - \frac{x}{y^2} re^t$$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t} = \frac{1}{y} rse^{st} - \frac{x}{y^2} rse^t$$

Implicit differentiation. We suppose that an equation of the form $F(x,y)=0$ defines y implicitly as a differentiable function of x, that is $y=f(x)$, where $F(x,f(x))=0$ for all x in the domain of f.

In order to find dy/dx, we differentiate both parts of the equation $F(x,y)=0$ with respect to x:

$$\frac{d}{dx}(F(x,y))=\frac{d}{dx}(0)$$

$$\frac{\partial F}{\partial x}\frac{dx}{dx}+\frac{\partial F}{\partial y}\frac{dy}{dx}=0$$

Since $\frac{dx}{dx}=1$, then

$$\boxed{\frac{dy}{dx}=-\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}}}$$

Example 5. Find dy/dx if $x^2-xy+y^3=8$.

$$F(x,y)=x^2-xy+y^3-8$$

$$\frac{\partial F}{\partial x}=2x-y$$

$$\frac{\partial F}{\partial y}=-x+3y^2$$

$$\frac{dy}{dx}=-\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}}=\boxed{-\frac{2x-y}{-x+3y^2}}$$

Suppose that z is given implicitly as a function $z=f(x,y)$ by an equation of the form $F(x,y,z)=0$. If F is differentiable and f_x and f_y exist, then we can use the Chain Rule to differentiate the equation $F(x,y,z)=0$ as follows

$$\frac{\partial F}{\partial x}\frac{\partial x}{\partial x}+\frac{\partial F}{\partial y}\frac{\partial y}{\partial x}+\frac{\partial F}{\partial x}\frac{\partial x}{\partial x}=0$$

Since $\frac{\partial x}{\partial x}=1$ and $\frac{\partial y}{\partial x}=0$, then

$$\frac{\partial F}{\partial x}+\frac{\partial F}{\partial z}\frac{\partial z}{\partial x}=0$$

or

$$\boxed{\frac{\partial z}{\partial x}=-\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial z}}}\qquad\boxed{\frac{\partial z}{\partial y}=-\frac{\frac{\partial F}{\partial y}}{\frac{\partial F}{\partial z}}}$$

Example 6. Find $\partial z/\partial x$ and $\partial z/\partial y$ if $xyz=\cos(x+y+z)$.

$$xyz-\cos(x+y+z)=0$$

$$F(x,y,z)=xyz-\cos(x+y+z)$$

$$\frac{\partial F}{\partial x}=yz+\sin(x+y+z)$$

$$\frac{\partial F}{\partial y}=xz+\sin(x+y+z)$$

$$\frac{\partial F}{\partial z}=xy+\sin(x+y+z)$$

$$\frac{\partial z}{\partial x}=-\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial z}}=-\frac{yz+\sin(x+y+z)}{xy+\sin(x+y+z)}$$

$$\frac{\partial z}{\partial y}=-\frac{\frac{\partial F}{\partial y}}{\frac{\partial F}{\partial z}}=-\frac{xz+\sin(x+y+z)}{xy+\sin(x+y+z)}$$