

Section 14.6 Directional derivatives and the gradient vector.

Let  $z = f(x, y)$ . We wish to find the rate of change of  $z$  at  $(x_0, y_0)$  in the direction of an arbitrary unit vector  $\mathbf{u} = \langle a, b \rangle$ .

- Consider the surface  $S$  with equation  $z = f(x, y)$  and we let  $z_0 = f(x_0, y_0)$ .
- The point  $P(x_0, y_0, z_0)$  lies on  $S$ .
- The vertical plane that passes through  $P$  in the direction  $\mathbf{u}$  intersects  $S$  in a curve  $C$ .
- The slope of the tangent line  $T$  to  $C$  at  $P$  is the rate of change of  $z$  in the direction of  $\mathbf{u}$ .
- Let  $Q(x, y, z)$  be another point on  $C$ . If  $P'(x_0, y_0, 0)$  and  $Q'(x, y, 0)$  are projections of  $P$  and  $Q$  on the  $xy$ -plane
- The vector  $\overrightarrow{P'Q'} = \langle x - x_0, y - y_0, 0 \rangle$  is parallel to  $\mathbf{u}$  and so

$$\overrightarrow{P'Q'} = h\mathbf{u} = \langle ha, hb \rangle$$

for some scalar  $h$

- Therefore  $x - x_0 = ha$   $y - y_0 = hb$  and  $\frac{\Delta z}{h} = \frac{z - z_0}{h} = \frac{f(x_0 + ha, y_0 + hb) - f(x_0, y_0)}{h}$
- If we take the limit as  $h \rightarrow 0$ , we obtain the rate of change of  $z$  in the direction of  $\mathbf{u}$ , which is called the directional derivative of  $f$  in the direction of  $\mathbf{u}$

**Definition.** The **directional derivative** of  $f$  at  $(x_0, y_0)$  in the direction of a unit vector  $\mathbf{u} = \langle a, b \rangle$  is

$$D_{\mathbf{u}}f(x_0, y_0) = \lim_{h \rightarrow 0} \frac{z - z_0}{h} = \lim_{h \rightarrow 0} \frac{f(x_0 + ha, y_0 + hb) - f(x_0, y_0)}{h}$$

if this limit exists.

**Theorem.** If  $f$  is a differentiable function of  $x$  and  $y$ , then  $f$  has a directional derivative in the direction of any unit vector  $\mathbf{u} = \langle a, b \rangle$  and

$$D_{\mathbf{u}}f(x, y) = \frac{\partial f}{\partial x}(x, y)a + \frac{\partial f}{\partial y}(x, y)b$$

If the unit vector  $\mathbf{u}$  makes an angle  $\theta$  with the positive  $x$ -axis, then

$$D_{\mathbf{u}}f(x, y) = \frac{\partial f}{\partial x}(x, y) \cos \theta + \frac{\partial f}{\partial y}(x, y) \sin \theta$$

**Example 1.** Find the directional derivative of the function  $f(x, y) = y^x$  at the point  $(1, 2)$  in the direction of the unit vector  $\mathbf{u}$  given by angle  $\theta = -2\pi/3$ .

**Definition.** If  $f$  is a function of two variables  $x$  and  $y$ , then the **gradient** of  $f$  is defined by

$$\text{grad}(f) = \nabla f(x, y) = \left\langle \frac{\partial f}{\partial x}(x, y), \frac{\partial f}{\partial y}(x, y) \right\rangle$$

Then

$$D_{\mathbf{u}}f(x, y) = \nabla f(x, y) \cdot \mathbf{u}$$

**Example 2.** Find the directional derivative of the function  $f(x, y) = xe^{xy}$  at the point  $(-3, 0)$  in the direction of the vector  $\mathbf{v} = 2\mathbf{i} + 3\mathbf{j}$ .

For a function of three variables  $w = f(x, y, z)$  the **gradient vector** is

$$\text{grad}(f) = \nabla f(x, y, z) = \left\langle \frac{\partial f}{\partial x}(x, y, z), \frac{\partial f}{\partial y}(x, y, z), \frac{\partial f}{\partial z}(x, y, z) \right\rangle$$

and

$$D_{\mathbf{u}}f(x, y, z) = \nabla f(x, y, z) \cdot \mathbf{u}$$

**Theorem.** Suppose  $f$  is a differentiable function of two or three variables and  $\vec{x} = \langle x, y \rangle$  if  $f$  is a function of two variables  $\vec{x} = \langle x, y, z \rangle$  if  $f$  is a function of three variables. The maximum value of the directional derivative  $D_{\mathbf{u}}f(\vec{x})$  is  $|\nabla f(\vec{x})|$  and it occurs when  $\mathbf{u}$  has the same direction as the gradient vector  $\nabla f(\vec{x})$ .

**Example 3.** Suppose that over a certain region of space the electrical potential  $V$  is given by  $V(x, y, z) = 5x^2 - 3xy + xyz$ .

1. Find the rate of change of the potential at  $P(3, 4, 5)$  in the direction of the vector  $\mathbf{v} = \langle 1, 1, -1 \rangle$ .

2. In which direction does  $V$  change most rapidly at  $P$ ?

3. What is the maximum rate of change at  $P$ ?

**Tangent planes to level surfaces.**

Suppose  $S$  is a surface with equation  $F(x, y, z) = k$ , that is, it is a level surface of the function  $w = F(x, y, z)$ , and let  $P(x_0, y_0, z_0)$  be a point on  $S$ .

We define the **tangent plane to the level surface**  $F(x, y, z) = k$  **at**  $P(x_0, y_0, z_0)$  as the plane that passes through  $P$  and has normal vector  $\nabla F(x_0, y_0, z_0)$  and its equation is

$$\boxed{F_x(x_0, y_0, z_0)(x - x_0) + F_y(x_0, y_0, z_0)(y - y_0) + F_z(x_0, y_0, z_0)(z - z_0) = 0}$$

The **normal line** to  $S$  at  $P$  is the line passing through  $P$  and perpendicular to the tangent plane (its direction is given by the gradient vector  $\nabla F(x_0, y_0, z_0)$ ). Its symmetric equations are

$$\boxed{\frac{x - x_0}{F_x(x_0, y_0, z_0)} = \frac{y - y_0}{F_y(x_0, y_0, z_0)} = \frac{z - z_0}{F_z(x_0, y_0, z_0)}}$$

If the equation of a surface  $S$  is of the form  $z = f(x, y)$ , we can rewrite

$$F(x, y, z) = f(x, y) - z = 0$$

and regard  $S$  as a level surface of  $F$  with  $k = 0$ . Then

$$F_x(x_0, y_0, z_0) = f_x(x_0, y_0), \quad F_y(x_0, y_0, z_0) = f_y(x_0, y_0), \quad F_z(x_0, y_0, z_0) = -1$$

so the equation of the tangent plane to  $S$  at  $(x_0, y_0)$  is

$$f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) - (z - z_0) = 0$$

and the equation of the normal line to  $S$  at  $(x_0, y_0)$  is

$$\boxed{\frac{x - x_0}{f_x(x_0, y_0)} = \frac{y - y_0}{f_y(x_0, y_0)} = \frac{z - z_0}{-1}}$$

**Example 4.** Find equations of the tangent plane and the normal line to the surface  $x^2 - 2y^2 - 3z^2 + xyz = 4$  at the point  $(3, -2, -1)$ .

**Example 5.** Find parametric equations for the tangent line to the curve of intersection of the paraboloid  $z = x^2 + y^2$  and the ellipsoid  $4x^2 + y^2 + z^2 = 9$  at the point  $(-1, 1, 2)$ .