Section 14.6 **Directional derivatives and the gradient vector.**

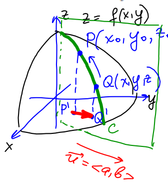
Let $z = f(x, y)$. We wish to find the rate of change of z at (x_0, y_0) in the direction of an arbitrary unit vector $\mathbf{u} = <a, b>$. (u is in the xy-plane)

- Consider the surface S with equation $z = f(x, y)$ and we let $z_0 = f(x_0, y_0)$.
- The point $P(x_0, y_0, z_0)$ lies on S.
- The vertical plane that passes through P in the direction $\mathbf{u}$ intersects S in a curve C.
- The slope of the tangent line T to C at P is the rate of change of z in the direction of $\mathbf{u}$.
- Let $Q(x, y, z)$ be another point on C. If $P'(x_0, y_0, 0)$ and $Q'(x, y, 0)$ are projections of P and Q on the xy-plane
- The vector $\overrightarrow{P'Q'} = <x - x_0, y - y_0, 0>$ is parallel to $\mathbf{u}$ and so

$$\overrightarrow{P'Q'} = h\mathbf{u} = <ha, hb>$$

for some scalar h

- Therefore $x - x_0 = ha \quad y - y_0 = hb$ and $\boxed{\frac{\Delta z}{h} = \frac{z - z_0}{h} = \frac{f(x_0 + ha, y_0 + hb) - f(x_0, y_0)}{h}}$
- If we take the limit as $h \to 0$, we obtain the rate of change of z in the direction of $\mathbf{u}$, which is called the directional derivative of f in the direction of $\mathbf{u}$



$P(x_0, y_0, z_0)$, where $z_0 = f(x_0, y_0)$

$P'(x_0, y_0, 0)$, $Q'(x, y, 0)$

$\overrightarrow{P'Q'} \parallel \vec{u}$

$\overrightarrow{P'Q'} = <x - x_0, y - y_0, 0>$

Definition. The directional derivative of f at (x_0, y_0) in the direction of a unit vector $\mathbf{u} = <a, b>$ is

$$D_{\mathbf{u}} f(x_0, y_0) = \lim_{h \to 0} \frac{z - z_0}{h} = \lim_{h \to 0} \frac{f(x_0 + ha, y_0 + hb) - f(x_0, y_0)}{h}$$

if this limit exists.

Theorem. If f is a differentiable function of x and y, then f has a directional derivative in the direction of any unit vector $\mathbf{u} = <a, b>$ and

$$\boxed{D_{\mathbf{u}} f(x, y) = \frac{\partial f}{\partial x}(x, y)a + \frac{\partial f}{\partial y}(x, y)b}$$

If the unit vector $\mathbf{u}$ makes an angle θ with the positive x-axis, then

$$\boxed{D_{\mathbf{u}} f(x, y) = \frac{\partial f}{\partial x}(x, y)\cos\theta + \frac{\partial f}{\partial y}(x, y)\sin\theta}$$

Example 1. Find the directional derivative of the function $f(x, y) = y^x$ at the point $(1, 2)$ in the direction of the unit vector $\mathbf{u}$ given by angle $\theta = -2\pi/3$.

$$D_{\vec{u}} f = \frac{\partial f}{\partial x}(1,2) \cos\Big(\underbrace{-\frac{2\pi}{3}}_{-1/2}\Big) + \frac{\partial f}{\partial y}(1,2) \sin\Big(\underbrace{-\frac{2\pi}{3}}_{-\frac{\sqrt{3}}{2}}\Big)$$

$\frac{\partial f}{\partial x} = y^x \ln y$ | $\frac{\partial f}{\partial x}(1,2) = 2\ln 2$

$\frac{\partial f}{\partial y} = xy^{x-1}$ | $\frac{\partial f}{\partial y}(1,2) = 1 \cdot 2^{1-1} = 1$

$$D_{\vec{u}} f = \boxed{2\ln 2\left(-\frac{1}{2}\right) - \frac{\sqrt{3}}{2}}$$

Definition. If f is a function of two variables x and y, then the gradient of f is defined by

$$\boxed{\text{grad}(f) = \nabla f(x, y) = \left\langle \frac{\partial f}{\partial x}(x, y), \frac{\partial f}{\partial y}(x, y) \right\rangle}$$

Then

$$\boxed{D_{\mathbf{u}} f(x, y) = \nabla f(x, y) \cdot \mathbf{u}}$$

Example 2. Find the directional derivative of the function $f(x, y) = xe^{xy}$ at the point $(-3, 0)$ in the direction of the vector $\mathbf{v} = 2\mathbf{i} + 3\mathbf{j}$. $= <2, 3>$

$|\vec{v}| = \sqrt{4+9} = \sqrt{13}$, $\quad \vec{u} = \frac{\vec{v}}{|\vec{v}|} = <\frac{2}{\sqrt{13}}, \frac{3}{\sqrt{13}}>$

$\text{grad } f = <\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}> = <e^{xy} + xe^{xy}(y), xe^{xy}(x)>$

$\text{grad } f(-3, 0) = <1, 9>$

$D_{\vec{u}} f = <1, 9> \cdot <\frac{2}{\sqrt{13}}, \frac{3}{\sqrt{13}}> = \frac{2}{\sqrt{13}} + \frac{27}{\sqrt{13}} = \boxed{\frac{29}{\sqrt{13}}}$

For a function of three variables $w = f(x, y, z)$ the **gradient vector** is

$$\boxed{\mathrm{grad}(f) = \nabla f(x, y, z) = \left\langle \frac{\partial f}{\partial x}(x, y, z), \frac{\partial f}{\partial y}(x, y, z), \frac{\partial f}{\partial z}(x, y, z) \right\rangle}$$

and

$$\boxed{D_{\mathbf{u}} f(x, y, z) = \nabla f(x, y, z) \cdot \mathbf{u}}$$

Theorem. Suppose f is a differentiable function of two or three variables and $\vec{v} = < x, y >$ if f is a function of two variables $\vec{v} = < x, y, z >$ if f is a function of three variables. The maximum value of the directional derivative $D_{\mathbf{u}} f(\vec{v})$ is $|\nabla f(\vec{v})|$ and it occurs when $\mathbf{u}$ has the same direction as the gradient vector $\nabla f(\vec{v})$.

Example 3. Suppose that over a certain region of space the electrical potential V is given by $V(x, y, z) = 5x^2 - 3xy + xyz$.

1. Find the rate of change of the potential at $P(3, 4, 5)$ in the direction of the vector $\mathbf{v} = < 1, 1, -1 >$.

$|\vec{v}| = \sqrt{3}$

$\vec{u} = \frac{\vec{v}}{|\vec{v}|} = < \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}} >$

$\mathrm{grad}\ V = < \frac{\partial V}{\partial x}, \frac{\partial V}{\partial y}, \frac{\partial V}{\partial z} > = < 10x - 3y + yz, -3x + xz, xy >$

$\mathrm{grad}\ V(3,4,5) = < 10(3) - 3(4) + 4(5), -3(3) + 3(5), 3(4) > = < 38, 6, 12 >$

$D_{\vec{u}} f = < 38, 6, 12 > \cdot < \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}} > = \frac{38 + 6 - 12}{\sqrt{3}} = \boxed{\frac{32}{\sqrt{3}}}$

2. In which direction does V change most rapidly at P?

$\mathrm{grad}\ V(3,4,5) = < 38, 6, 12 >$

3. What is the maximum rate of change at P?

$|\mathrm{grad}\, V(3,4,5)| = \sqrt{38^2 + 6^2 + 12^2} = \sqrt{1624}$

Tangent planes to level surfaces.

Suppose S is a surface with equation $F(x, y, z) = k$, that is, it is a level surface of the function $w = F(x, y, z)$, and let $P(x_0, y_0, z_0)$ be a point on S.

We define the tangent plane to the level surface $F(x, y, z) = k$ at $P(x_0, y_0, z_0)$ as the plane that passes through P and has normal vector $\nabla F(x_0, y_0, z_0)$ and its equation is

$$\boxed{F_x(x_0, y_0, z_0)(x - x_0) + F_y(x_0, y_0, z_0)(y - y_0) + F_z(x_0, y_0, z_0)(z - z_0) = 0}$$

The normal line to S at P is the line passing through P and perpendicular to the tangent plane (its direction is given by the gradient vector $\nabla F(x_0, y_0, z_0)$). Its symmetric equations are

$$\boxed{\frac{x - x_0}{F_x(x_0, y_0, z_0)} = \frac{y - y_0}{F_y(x_0, y_0, z_0)} = \frac{z - z_0}{F_z(x_0, y_0, z_0)}}$$

If the equation of a surface S is of the form $z = f(x, y)$, we can rewrite

$$F(x, y, z) = f(x, y) - z = 0$$

and regard S as a level surface of F with $k = 0$. Then

$$F_x(x_0, y_0, z_0) = f_x(x_0, y_0), \quad F_y(x_0, y_0, z_0) = f_y(x_0, y_0), \quad F_z(x_0, y_0, z_0) = -1$$

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so the equation of the tangent plane to S at (x_0, y_0) is $S: z = f(x,y)$

$$\boxed{f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) - (z - z_0) = 0}$$

and the equation of the normal line to S at (x_0, y_0) is

$$\boxed{\frac{x - x_0}{f_x(x_0, y_0)} = \frac{y - y_0}{f_y(x_0, y_0)} = \frac{z - z_0}{-1}}$$

Example 4. Find equations of the tangent plane and the normal line to the surface $x^2 - 2y^2 - 3z^2 + xyz = 4$ at the point $(3, -2, -1)$.

$F(x,y,z) = x^2 - 2y^2 - 3z^2 + xyz - 4$

grad $F = \nabla F = \langle \frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z} \rangle = \langle 2x + yz, -4y + xz, -6z + xy \rangle$

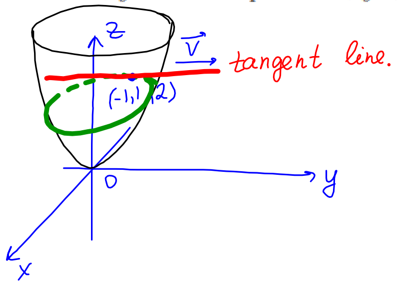
$\nabla F(3,-2,-1) = \langle 2(3) + (-2)(-1), -4(-2) + 3(-1), -6(-1) + 3(-2) \rangle$

$= \langle 8, 5, 0 \rangle$

Tangent plane: $\boxed{8(x-3) + 5(y+2) + 0(z+1) = 0}$

Normal line: $\boxed{\begin{cases} x = 3 + 8t \\ y = -2 + 5t \\ z = -1 \end{cases}}$

Example 5. Find parametric equations for the tangent line to the curve of intersection of the paraboloid $z = x^2 + y^2$ and the ellipsoid $4x^2 + y^2 + z^2 = 9$ at the point $(-1, 1, 2)$.



$f(x,y,z) = z - x^2 - y^2$

$g(x,y,z) = 4x^2 + y^2 + z^2 - 9$

The tangent line is going to be perpendicular to both $\nabla f(-1,1,2)$ and $\nabla g(-1,1,2)$

$\vec{V} = \nabla f(-1,1,2) \times \nabla g(-1,1,2)$

$\nabla f = \langle -2x, -2y, 1 \rangle$, $\nabla f(-1,1,2) = \langle 2, -2, 1 \rangle$

$\nabla g = \langle 8x, 2y, 2z \rangle$, $\nabla g(-1,1,2) = \langle -8, 2, 4 \rangle$

$\vec{V} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & -2 & 1 \\ -8 & 2 & 4 \end{vmatrix} = \vec{i}(-8-2) - \vec{j}(8+8) + \vec{k}(4-16)$

$= -10\vec{i} - 16\vec{j} - 12\vec{k}$

Parametric equations: $\boxed{\begin{cases} x = -1 - 10t \\ y = 1 - 16t \\ z = 2 - 12t \end{cases}}$