Section 14.7 **Maximum and minimum values.**

Definition. A function of two variables has a **local maximum** at (a, b) if $f(x, y) \leq f(a, b)$ for all points (x, y) in some disk with center (a, b). The number $f(a, b)$ is called a **local maximum value**. If $f(x, y) \geq f(x, y)$ for all (x, y) in such a disk, $f(a, b)$ is a **local minimum value**.

Theorem. If f has a local extremum (that is, a local maximum or minimum) at (a, b) and the first-order partial derivatives of f exist there, then

$$\boxed{f_x(a, b) = f_y(a, b) = 0}$$

Geometric interpretation of the Theorem: if the graph of f has a tangent plane at a local extremum, then the tangent plane must be horizontal.

A point (a, b) such that $f_x(a, b) = f_y(a, b) = 0$ or one of these partial derivatives does not exist, is called a **critical point** of f. At a critical point, a function could have a local minimum or a local maximum or neither.

Example 1. Find the extreme values of $f(x, y) = y^2 - x^2$.

Second derivative test. Suppose the second partial derivatives of f are continuous in a disk with center (a, b), and suppose $f_x(a, b) = f_y(a, b) = 0$. Let

$$\boxed{D = D(a, b) = \begin{vmatrix} f_{xx}(a, b) & f_{xy}(a, b) \\ f_{xy}(a, b) & f_{yy}(a, b) \end{vmatrix} = f_{xx}(a, b) f_{yy}(a, b) - [f_{xy}(a, b)]^2}$$

(a) If $D > 0$ and $f_{xx}(a, b) > 0$, then $f(a, b)$ is a local minimum.
(b) If $D > 0$ and $f_{xx}(a, b) < 0$, then $f(a, b)$ is a local maximum.
(c) If $D < 0$, then $f(a, b)$ is not a local extremum. $f(a, b)$ is a **saddle point**.
If $D = 0$ the test gives no information.

Example 2. Find the local extrema of $f(x, y) = x^3 - 3xy + y^3$.

Example 3. Find the points on the surface $z^2 = xy + 1$ that are closest to the origin.

Absolute maximum and minimum values. A **closed set** in $\mathbb{R}^2$ is one that contains all its boundary points. A **bounded set** in $\mathbb{R}^2$ is one that contained within some disk.

Extreme value theorem for functions of two variables. If f is continuous on a closed, bounded set D in $\mathbb{R}^2$, then f attains an absolute maximum value $f(x_1, y_1)$ and an absolute minimum value $f(x_2, y_2)$ at some points (x_1, y_1) and (x_2, y_2) in D.

To find the absolute maximum and minimum values of a continuous function f on a closed bounded set D:

1. Find the values of f at the critical points of f in D.
2. Find the extreme values of f on the boundary of D.
3. The largest of the values of f from steps 1 and 2 is the absolute maximum value; the smallest of these values is the absolute minimum value.

Example 4. Find the absolute maximum and minimum values of the function $f(x, y) = x^2 + y^2 + x^2y + 4$ on the set $D = \{(x, y) | |x| \leq 1, |y| \leq 1\}$.