Section 14.7 **Maximum and minimum values.**

Definition. A function of two variables has a local maximum at (a, b) if $f(x, y) \le f(a, b)$ for all points (x, y) in some disk with center (a, b). The number $f(a, b)$ is called a local maximum value. If $f(x, y) \ge f(a, b)$ for all (x, y) in such a disk, $f(a, b)$ is a local minimum value.

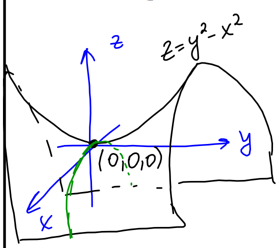
Theorem. If f has a local extremum (that is, a local maximum or minimum) at (a, b) and the first-order partial derivatives of f exist there, then

$$\boxed{f_x(a, b) = f_y(a, b) = 0}$$

Geometric interpretation of the Theorem: if the graph of f has a tangent plane at a local extremum, then the tangent plane must be horizontal.

A point (a, b) such that $f_x(a, b) = f_y(a, b) = 0$ or one of these partial derivatives does not exist, is called a **critical point** of f. At a critical point, a function could have a local minimum or a local maximum or neither.

Example 1. Find the extreme values of $f(x, y) = y^2 - x^2$.



$f_x(x,y) = -2x = 0 \Rightarrow x = 0$

$f_y(x,y) = 2y = 0 \Rightarrow y = 0$

$f(0,0) = 0$

Point $(0,0)$

$f(x,0) = -x^2 \le 0$, $f(x,0) \le f(0,0)$ for all x

$f(0,y) = y^2 \ge 0$ $f(0,y) \ge f(0,0)$ for all y

$(0,0)$ is a saddle point.

Second derivative test. Suppose the second partial derivatives of f are continuous in a disk with center (a, b), and suppose $f_x(a, b) = f_y(a, b) = 0$. Let

$$\boxed{D = D(a, b) = \begin{vmatrix} f_{xx}(a, b) & f_{xy}(a, b) \\ f_{xy}(a, b) & f_{yy}(a, b) \end{vmatrix} = f_{xx}(a, b) f_{yy}(a, b) - [f_{xy}(a, b)]^2}$$

(a) If $D > 0$ and $f_{xx}(a, b) > 0$, then $f(a, b)$ is a local minimum.
(b) If $D > 0$ and $f_{xx}(a, b) < 0$, then $f(a, b)$ is a local maximum.
(c) If $D < 0$, then $f(a, b)$ is not a local extremum. $f(a, b)$ is a saddle point.
If $D = 0$ the test gives no information.

Example 2. Find the local extrema of $f(x, y) = x^3 - 3xy + y^3$.

$f_x = 3x^2 - 3y = 0$
$f_y = -3x + 3y^2 = 0$

$\begin{cases} x^2 = y \\ x = y^2 \end{cases}$

$x = (x^2)^2$ or $x = x^4$

$x - x^4 = 0$

$x(1 - x^3) = 0$

$x = 0$ or $x = 1$

$y = 0^2 = 0$ $\quad y = 1^2 = 1$

$f_{xx} = 6x$
$f_{xy} = -3$
$f_{yy} = 6y$

Critical points $(0,0)$ and $(1,1)$.

$D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{vmatrix} = \begin{vmatrix} 6x & -3 \\ -3 & 6y \end{vmatrix} = 36xy - 9$

$D(0,0) = -9 < 0$ $\quad \boxed{(0,0) \text{ saddle point}}$

$D(1,1) = 36 - 9 = 27 > 0$
$f_{xx}(1,1) = 6 > 0$ $\quad \boxed{(1,1) \text{ local min}}$

Example 3. Find the points on the surface $z^2 = xy + 1$ that are closest to the origin.

(x, y, z) is an arbitrary point on the surface.

Distance from (x, y, z) to $(0,0,0)$ is

$$d^2 = x^2 + y^2 + z^2$$

$$f(x,y) = d^2 = x^2 + y^2 + xy + 1$$

$$\frac{\partial f}{\partial x} = 2x + y = 0 \qquad y = -2x$$

$$\frac{\partial f}{\partial y} = 2y + x = 0 \qquad -3x = 0 \text{ or } x = 0 \text{ and } y = 0.$$

$(0,0)$ is a critical point.

$$D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{vmatrix} = \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} = 4 - 1 = 3 > 0$$

$$f_{xx}(0,0) = 2 > 0$$

$(0,0)$ local min for $x^2 + y^2 + xy + 1$

$z^2 = xy + 1$ if $x = y = 0$, then $z^2 = 1$

$z = \pm 1$

$$\boxed{(0,0,1) \text{ and } (0,0,-1)}$$

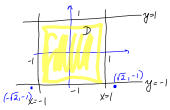
Absolute maximum and minimum values. A **closed set** in $\mathbb{R}^2$ is one that contains all its boundary points. A **bounded set** in $\mathbb{R}^2$ is one that contained within some disk.

Extreme value theorem for functions of two variables. If f is continuous on a closed, bounded set D in $\mathbb{R}^2$, then f attains an absolute maximum value $f(x_1, y_1)$ and an absolute minimum value $f(x_2, y_2)$ at some points (x_1, y_1) and (x_2, y_2) in D.

To find the absolute maximum and minimum values of a continuous function f on a closed bounded set D:

1. Find the values of f at the critical points of f in D.
2. Find the extreme values of f on the boundary of D.
3. The largest of the values of f from steps 1 and 2 is the absolute maximum value; the smallest of these values is the absolute minimum value.

Example 4. Find the absolute maximum and minimum values of the function $f(x, y) = x^2 + y^2 + x^2y + 4$ on the set $D = \{(x, y) | |x| \le 1, |y| \le 1\}$.



Critical points of f inside D.

$$\left.\begin{aligned} f_x &= 2x + 2xy = 0 \\ f_y &= 2y + x^2 = 0 \end{aligned}\right\} \Rightarrow 2x(1+y) = 0$$

$x = 0$ or $y = -1$

Plug $x = 0$ into the 2nd equation:

$2y = 0$ or $y = 0$

$\boxed{\text{Point } (0,0)}$

Plug $y = -1$ into the 2nd equation:

$x^2 = 2$ or $x = \pm\sqrt{2}$

Points $(\pm\sqrt{2}, -1)$ - not in D.

Critical points on the boundary of D:

line $x = 1$. $f(1, y) = 1 + y^2 + y + 4 = y^2 + y + 5$

$\frac{\partial f(1,y)}{\partial y} = 2y + 1 = 0$, $y = -1/2$

$\boxed{\text{Point } (1, -1/2)}$

line $x = -1$ $f(-1, y) = 1 + y^2 + y + 4 = y^2 + y + 5$

$\boxed{\text{Point } (-1, -1/2)}$

line $y = 1$ $f(x, 1) = x^2 + 1 + x^2 + 4 = 2x^2 + 5$

$\frac{\partial f(x,1)}{\partial x} = 4x = 0$

$\boxed{\text{Point } (0,1)}$

line $y = -1$ $f(x, -1) = x^2 + 1 - x^2 + 4 = 5$

no critical points.

$f(0,0) = \boxed{4 \text{ abs min value}}$

$f(1, -1/2) = 1 + \frac{1}{4} - \frac{1}{2} + 4 = 5 - \frac{1}{4} = \frac{19}{4}$

$f(-1, -1/2) = \frac{19}{4}$

$f(0,1) = 0 + 1 + 0 + 4 = \boxed{5 \text{ abs max value}}$