Section 14.8 **Lagrange multipliers.**

Problem. Find the extreme values of the function $w = f(x, y, z)$ subject to the constraint $g(x, y, z) = k$.
To find the maximum and minimum values of $f(x, y, z)$ subject to constraint $g(x, y, z) = k$ (assuming that these extreme values exist):

1. Find all values of x, y, z, and λ such that
$$\nabla f(x, y, z) = \lambda \nabla g(x, y, z)$$
and
$$g(x, y, z) = k$$
2. Evaluate f at all the points (x, y, z) that arise from step 1. The largest of these values is the maximum value of f; the smallest is the minimum value of f.

The number λ is called a **Lagrange multiplier**. The procedure described above is called the **method of Lagrange multipliers**.

If we rewrite the vector equation $\nabla f = \lambda \nabla g$ in terms of its components, then the equation in step 1 become

$$\boxed{\begin{aligned} f_x &= \lambda g_x \\ f_y &= \lambda g_y \\ f_z &= \lambda g_z \\ g(x, y, z) &= k \end{aligned}}$$

Example 1. Use Lagrange multipliers to find the maximum and minimum values of the function $f(x, y) = x^2 + y^2$ subject to the constraint $x^4 + y^4 = 1$.

$f(x,y) = x^2+y^2$, $g(x,y) = x^4+y^4$

grad $f = \nabla f = < \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} > = <2x, 2y>$

grad $g = \nabla g = < \frac{\partial g}{\partial x}, \frac{\partial g}{\partial y} > = <4x^3, 4y^3>$

grad $f = \lambda$(grad g)

$<2x, 2y> = \lambda <4x^3, 4y^3>$

component form: $\begin{cases} 2x = 4\lambda x^3 \\ 2y = 4\lambda y^3 \\ x^4+y^4=1 \end{cases}$

$x = 2\lambda x^3$
$2\lambda x^3 - x = 0$
$x(2\lambda x^2 - 1) = 0$
$x = 0$ or $x^2 = \frac{1}{2\lambda}$
$x = \pm\frac{1}{\sqrt{2\lambda}}, \lambda > 0$

$y = 2\lambda y^3$ or $(2\lambda y^2 - 1)y = 0$
$y = 0$ or $2\lambda y^2 - 1 = 0$
$y = \pm\frac{1}{\sqrt{2\lambda}}, \lambda > 0$

$x^4+y^4=1$

Plug $x=0$: $y^4 = 1$ or $y = \pm 1$
$\boxed{\text{Points } (0, \pm 1)}$

Plug $y=0$: $x^4 = 1$ or $x = \pm 1$
$\boxed{\text{Points } (\pm 1, 0)}$

Plug $x = \pm\frac{1}{\sqrt{2\lambda}}, y = \pm\frac{1}{\sqrt{2\lambda}}$: $\frac{1}{4\lambda^2} + \frac{1}{4\lambda^2} = 1$
$\frac{1}{2\lambda^2} = 1$ or $2\lambda^2 = 1$
$\lambda^2 = 1/2, \lambda = \frac{1}{\sqrt{2}}$

$x = \pm\frac{1}{\sqrt{2 \cdot \frac{1}{\sqrt{2}}}} = \pm\frac{1}{\sqrt[4]{2}}$, $y = \pm\frac{1}{\sqrt[4]{2}}$

$\boxed{\text{Points } \left(\frac{\pm 1}{\sqrt[4]{2}}, \frac{\pm 1}{\sqrt[4]{2}}\right)}$

Evaluat $f(x,y)$ @ points:
$f(0, \pm 1) = \boxed{1}$, min $f(\pm 1, 0) = 1$, $f\left(\frac{\pm 1}{\sqrt[4]{2}}, \pm\frac{1}{\sqrt[4]{2}}\right) = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = \boxed{\sqrt{2}}$ max

Suppose now we want to find the maximum and minimum values of $f(x, y, z)$ subject to two constrains (side conditions) of the form $g(x, y, z) = k$ and $h(x, y, z) = c$. There are numbers λ and μ (called Lagrange multipliers) such that

$$\nabla f(x, y, z) = \lambda \nabla g(x, y, z) + \mu \nabla h(x, y, z)$$

In this case Lagrange's method is to look for extreme values by solving five equations

$$\boxed{\begin{array}{l} f_x = \lambda g_x + \mu h_x \\ f_y = \lambda g_y + \mu h_y \\ f_z = \lambda g_z + \mu h_z \\ g(x, y, z) = k \\ h(x, y, z) = c \end{array}}$$

in unknowns x, y, z, λ, and μ.

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