Chapter 15. **Multiple integrals.**

Section 15.1 **Double integrals over rectangles.**

We would like to define the double integral of a function f of two variables that is defined on a closed rectangle

$$R = [a, b] \times [c, d] = \{(x, y) \in \mathbb{R}^2 | a \leq x \leq b, c \leq y \leq d\}$$

We take a partition P of R into subrectangles. This is accomplished by partitioning the intervals $[a, b]$ and $[c, d]$ as follows:

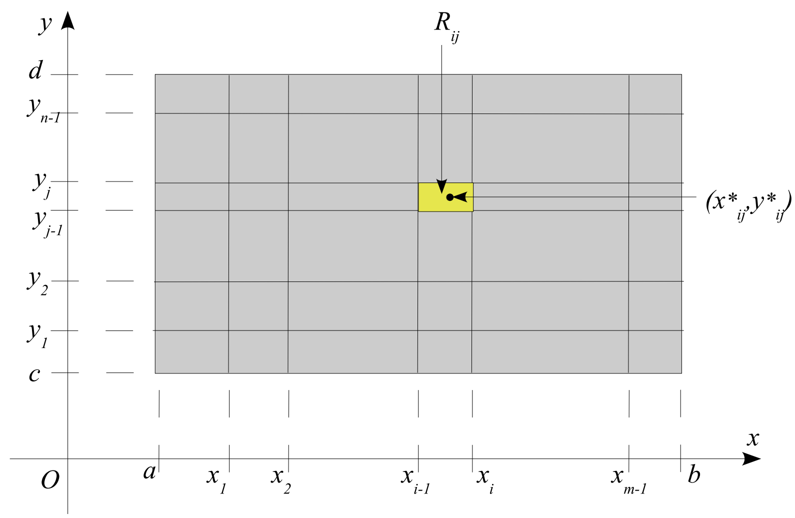
$$a = x_0 < x_1 < ... < x_{m-1} < x_m = b$$

$$c = y_0 < y_1 < ... < y_{n-1} < y_n = d$$

By drawing lines parallel to the coordinate axes through these partition points we form the subrectangles

$$R_{ij} = [x_{i-1}, x_i] \times [y_{j-1}, y_j]$$

for $i = 1, 2, ..., m$ and $j = 1, 2, ..., n$. There are mn of these subrectangles. If we let $\Delta x_i = x_i - x_{i-1}$ and $\Delta y_j = y_j - y_{j-1}$, then the area of R_{ij} is $\Delta A_{ij} = \Delta x_i \Delta y_j$.



Next we choose a point $(x^*_{ij}, y^*_{ij}) \in R_{ij}$ and form the **double Riemann sum**

$$\sum_{i=1}^{m} \sum_{j=1}^{n} f(x^*_{ij}, y^*_{ij}) \Delta A_{ij}$$

We denote by $\|P\|$ the **norm** of the partition, which is the length of the longest diagonal of all the subrectangles R_{ij}.

Definition. The **double integral** of f over the rectangle R is

$$\iint_R f(x, y) dA = \lim_{\|P\| \to 0} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x^*_{ij}, y^*_{ij}) \Delta A_{ij}$$

if the limit exists.

Note 1. In view of the fact that $\Delta A_{ij} = \Delta x_i \Delta y_j$, another notation that is used sometimes for the double integral is

$$\iint_R f(x, y) dA = \iint_R f(x, y) dx\ dy$$

Note 2. A function f is called **integrable** if the limit in the definition exists.

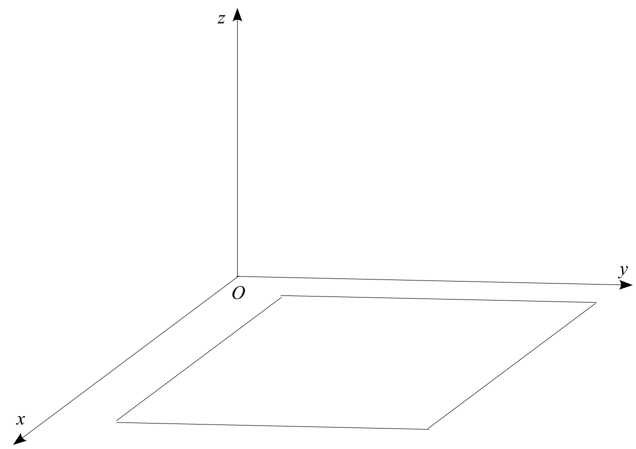
Example 1. Find an approximation to the integral

$$\iint\limits_R (x - 3y^2)dA$$

where $R = [0, 2] \times [1, 2]$, by computing the double Riemann sum with partition lines $x = 1$ and $y = 3/2$ and taking (x^*_{ij}, y^*_{ij}) to be the center of each rectangle.

Double integrals of positive functions can be interprepreted as volumes. Suppose that $f(x, y) \geq 0$ and f is defined on the rectangle $R = [a, b] \times [c, d]$. The graph of f is a surface with equation $z = f(a, b)$. Let S be the solid that lies above R and under the graph of f

$$S = \{(x, y, z) \in \mathbb{R}^3 | 0 \leq z \leq f(x, y), (x, y) \in R\}$$



If we partition R into subrectangles R_{ij} and choose (x_{ij}^*, y_{ij}^*) in R_{ij}, then we can approximate the part of S that lies above R_{ij} by a thin rectangular column with base R_{ij} and height $f(x_{ij}^*, y_{ij}^*)$. The volume of the column is

$$V_{ij} = f(x_{ij}^*, y_{ij}^*)\Delta A_{ij}$$

If we follow this procedure for all rectangles and add the volumes of the corresponding boxes, we get an approximation to the total volume of S

$$V = \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\Delta A_{ij}$$

Approximation becomes better if we use a finer partition P.

Theorem. If $f(x, y) \geq 0$ and f is continuous on the rectangle R, then the volume V of the solid that lie above R and under the surface $z = f(x, y)$ is

$$V = \iint\limits_R f(x, y)\ dA$$

Example 2. Evaluate the double integral

$$\iint\limits_R (8 - 4y) dA,$$

where $R = [0, 1] \times [0, 1]$ by first identifying it as the volume of a solid.

Iterated integrals.

Suppose f is a function of two variables that is integrable over the rectangle $R = [a, b] \times [c, d]$.

We use notation $\int\limits_c^d f(x, y)\ dy$ to mean that x is held fixed and $f(x, y)$ is integrated with respect to y from $y = c$ to $y = d$. This procedure is called **partial integration with respect to** y.

$$A(x) = \int\limits_c^d f(x, y)\ dy$$

$$\int\limits_a^b A(x)\ dx = \int\limits_a^b \left[\int\limits_c^d f(x, y)\ dy\right] dx$$

The integral $\int\limits_a^b \left[\int\limits_c^d f(x, y)\ dy\right] dx$ is called an **iterated integral**. Thus,

$$\int\limits_a^b \int\limits_c^d f(x, y) dy dx = \int\limits_a^b \left[\int\limits_c^d f(x, y)\ dy\right] dx$$

means that we first integrate with respect to y from c to d and then with respect to x from a to b.

Similarly, the iterated integral

$$\int\limits_c^d \int\limits_a^b f(x, y) dx dy = \int\limits_c^d \left[\int\limits_a^b f(x, y)\ dx\right] dy$$

means that we first integrate with respect to x from a to b and then with respect to y from c to d.

Example 1. Evaluate the iterated integrals:

1. $\int\limits_0^3 \int\limits_0^1 \sqrt{x + y}\ dx dy$

2. $\int_0^1 \int_0^1 \frac{xy}{\sqrt{x^2+y^2+1}} dydx$

Fubini's Theorem. If f is continuous on the rectangle $R = [a, b] \times [c, d]$, then

$$\iint_R f(x,y)\ dA = \int_c^d \int_a^b f(x,y) dxdy = \int_a^b \int_c^d f(x,y) dydx$$

Example 2. Calculate the double integral

$$\iint_R \left(xy^2 + \frac{y}{x} \right) dA,$$

where $R = \{(x, y)|2 \leq x \leq 3, -1 \leq y \leq 0\}$.

Example 3. Find the volume of the solid lying under the elliptic paraboloid $\dfrac{x^2}{4} + \dfrac{y^2}{9} + z = 1$ and above the rectangle $R = [-1, 1] \times [-2, 2]$.