Chapter 15. **Multiple integrals.**
Section 15.1 **Double integrals over rectangles.**

We would like to define the double integral of a function f of two variables that is defined on a closed rectangle

$$R = [a,b] \times [c,d] = \{(x,y) \in \mathbb{R}^2 | a \le x \le b, c \le y \le d\}$$

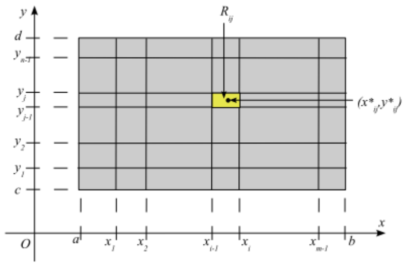
We take a partition P of R into subrectangles. This is accomplished by partitioning the intervals $[a,b]$ and $[c,d]$ as follows:

$$a = x_0 < x_1 < ... < x_{m-1} < x_m = b$$
$$c = y_0 < y_1 < ... < y_{n-1} < y_n = d$$

By drawing lines parallel to the coordinate axes through these partition points we form the subrectangles

$$R_{ij} = [x_{i-1}, x_i] \times [y_{j-1}, y_j]$$

for $i = 1, 2, ..., m$ and $j = 1, 2, ..., n$. There are mn of these subrectangles. If we let $\Delta x_i = x_i - x_{i-1}$ and $\Delta y_j = y_j - y_{j-1}$, then the area of R_{ij} is $\Delta A_{ij} = \Delta x_i \Delta y_j$.



Next we choose a point $(x^*_{ij}, y^*_{ij}) \in R_{ij}$ and form the **double Riemann sum**

$$\sum_{i=1}^{m}\sum_{j=1}^{n} f(x^*_{ij}, y^*_{ij})\Delta A_{ij}$$

We denote by $\|P\|$ the **norm** of the partition, which is the length of the longest diagonal of all the subrectangles R_{ij}.

Definition. The **double integral** of f over the rectangle R is

$$\boxed{\iint_R f(x,y)dA = \lim_{\|P\|\to 0} \sum_{i=1}^{m}\sum_{j=1}^{n} f(x^*_{ij}, y^*_{ij})\Delta A_{ij}}$$

if the limit exists.

Note 1. In view of the fact that $\Delta A_{ij} = \Delta x_i \Delta y_j$, another notation that is used sometimes for the double integral is

$$\iint_R f(x,y)dA = \iint_R f(x,y)dx\,dy$$

Note 2. A function f is called **integrable** if the limit in the definition exists.

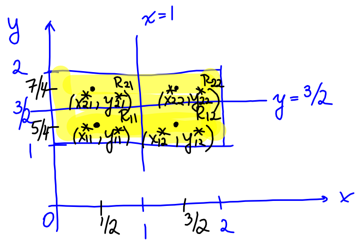
Example 1. Find an approximation to the integral

$$\iint_R (x - 3y^2)dA$$

where $R = \overset{x}{[0,2]} \times \overset{y}{[1,2]}$, by computing the double Riemann sum with partition lines $x = 1$ and $y = 3/2$ and taking (x^*_{ij}, y^*_{ij}) to be the center of each rectangle.

$R: \quad 0 \le x \le 2, \quad 1 \le y \le 2$

$A(R_{11}) = A(R_{12}) = A(R_{21}) = A(R_{22}) = \frac{1}{2}(1) = \frac{1}{2}$



$(x^*_{11}, y^*_{11}) = (\frac{1}{2}, \frac{5}{4})$	$f(x^*_{11}, y^*_{11}) = \frac{1}{2} - 3\cdot(\frac{5}{4})^2$
$(x^*_{12}, y^*_{12}) = (\frac{3}{2}, \frac{5}{4})$	$f(x^*_{12}, y^*_{12}) = \frac{3}{2} - 3\cdot(\frac{5}{4})^2$
$(x^*_{21}, y^*_{21}) = (\frac{1}{2}, \frac{7}{4})$	$f(x^*_{21}, y^*_{21}) = \frac{1}{2} - 3(\frac{7}{4})^2$
$(x^*_{22}, y^*_{22}) = (\frac{3}{2}, \frac{7}{4})$	$f(x^*_{22}, y^*_{22}) = \frac{3}{2} - 3(\frac{7}{4})^2$

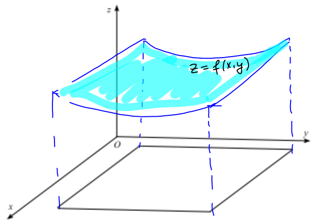
$$\iint_R (x-3y^2)dA \approx f(x^*_{11}, y^*_{11})A(R_{11}) + f(x^*_{12}, y^*_{12})A(R_{12}) + f(x^*_{21}, y^*_{21})A(R_{21}) + f(x^*_{22}, y^*_{22})A(R_{22})$$

$$= \frac{1}{2}\left[\frac{1}{2} - 3\left(\frac{5}{4}\right)^2 + \frac{3}{2} - 3\left(\frac{5}{4}\right)^2 + \frac{1}{2} - 3\left(\frac{7}{4}\right)^2 + \frac{3}{2} - 3\left(\frac{7}{4}\right)^2\right]$$

$$= \frac{1}{2}\left[3 - \frac{444}{16}\right] = \frac{1}{2}\cdot\left(-\frac{396}{16}\right) = \boxed{-\frac{99}{8}}$$

Double integrals of positive functions can be interprepreted as volumes. Suppose that $f(x,y) \geq 0$ and f is defined on the rectangle $R = [a,b] \times [c,d]$. The graph of f is a surface with equation $z = f(x,y)$. Let S be the solid that lies above R and under the graph of f

$$S = \{(x,y,z) \in \mathbb{R}^3 | 0 \leq z \leq f(x,y), (x,y) \in R\}$$



If we partition R into subrectangles R_{ij} and choose (x_{ij}^*, y_{ij}^*) in R_{ij}, then we can approximate the part of S that lies above R_{ij} by a thin rectangular column with base R_{ij} and height $f(x_{ij}^*, y_{ij}^*)$. The volume of the column is

$$V_{ij} = f(x_{ij}^*, y_{ij}^*)\Delta A_{ij}$$

If we follow this procedure for all rectangles and add the volumes of the corresponding boxes, we get an approximation to the total volume of S

$$V = \sum_{i=1}^{m}\sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\Delta A_{ij}$$

Approximation becomes better if we use a finer partition P.
Theorem. If $f(x,y) \geq 0$ and f is continuous on the rectangle R, then the volume V of the solid that lie above R and under the surface $z = f(x,y)$ is

$$\boxed{V = \iint_R f(x,y)\; dA}$$

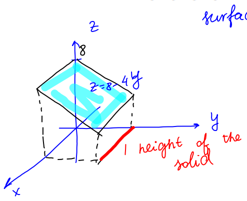
Example 2. Evaluate the double integral

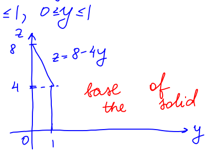
$$\iint_R (8-4y)dA, = \left[4(1) + \frac{1}{2}(4)(1)\right](1) = \boxed{6}$$

where $R = [0,1] \times [0,1]$ by first identifying it as the volume of a solid.

surface $z = 8-4y$ - plane

$0 \leq x \leq 1$, $0 \leq y \leq 1$





Iterated integrals.

Suppose f is a function of two variables that is integrable over the rectangle $R = [a,b] \times [c,d]$.

We use notation $\int_c^d f(x,y)\; dy$ to mean that x is held fixed and $f(x,y)$ is integrated with respect to y from $y = c$ to $y = d$. This procedure is called **partial integration with respect to** y.

$$A(x) = \int_c^d f(x,y)\; dy$$

$$\int_a^b A(x)\; dx = \int_a^b \left[\int_c^d f(x,y)\; dy\right] dx$$

The integral $\int_a^b \left[\int_c^d f(x,y)\; dy\right] dx$ is called an **iterated integral**. Thus,

$$\int_a^b \int_c^d f(x,y) dy dx = \int_a^b \left[\int_c^d f(x,y)\; dy\right] dx$$

means that we first integrate with respect to y from c to d and then with respect to x from a to b.

Similarly, the iterated integral

$$\int_c^d \int_a^b f(x,y) dx dy = \int_c^d \left[\int_a^b f(x,y)\; dx\right] dy$$

means that we first integrate with respect to x from a to b and then with respect to y from c to d.

Example 1. Evaluate the iterated integrals:

1. $\int_0^3 \left[\int_0^1 \sqrt{x+y}\,dx\right] dy = \int_0^3 \left[\frac{(x+y)^{3/2}}{3/2} \Big|_{x=0}^{x=1} \right] dy = \frac{2}{3}\int_0^3 \left[(1+y)^{3/2} - y^{3/2}\right] dy$

$$= \frac{2}{3}\left[\frac{(1+y)^{5/2}}{5/2} - \frac{y^{5/2}}{5/2}\right]_0^3 = \frac{4}{15}\left[4^{5/2} - 3^{5/2} - 1^{5/2} + 0\right]$$

$$= \boxed{\frac{4}{15}\left(31 - 3^{5/2}\right)}$$

2. $\int_0^1 \left[\int_0^1 \frac{xy}{\sqrt{x^2+y^2+1}}\,dy\right] dx =$

$u = x^2+y^2+1$

$du = \frac{\partial u}{\partial y}\,dy = 2y\,dy$

$y=0 \Rightarrow u = x^2+1$

$y=1 \Rightarrow u = x^2+1^2+1 = x^2+2$

$$= \int_0^1 x\left[\int_{x^2+1}^{x^2+2} \frac{du}{2\sqrt{u}}\right] dx = \frac{1}{2}\int_0^1 x \left[\frac{u^{1/2}}{1/2}\right]_{u=x^2+1}^{u=x^2+2} dx$$

$$= \int_0^1 x\left[\sqrt{x^2+2} - \sqrt{x^2+1}\right] dx = \int_0^1 x\sqrt{x^2+2}\,dx - \int_0^1 x\sqrt{x^2+1}\,dx$$

$v = x^2+2$, $dv = 2x\,dx$, $x=0 \Rightarrow v=2$, $x=1 \Rightarrow v=3$

$w = x^2+1$, $dw = 2x\,dx$, $x=0 \Rightarrow w=1$, $x=1 \Rightarrow w=2$

$$= \frac{1}{2}\int_2^3 \sqrt{v}\,dv - \frac{1}{2}\int_1^2 \sqrt{w}\,dw = \frac{1}{2}\left[\frac{v^{3/2}}{3/2}\Big|_2^3 - \frac{w^{3/2}}{3/2}\Big|_1^2\right]$$

$$= \frac{1}{3}\left(3\sqrt{3} - 2\sqrt{2} - 2\sqrt{2} + 1\right) = \boxed{\frac{1}{3}\left(3\sqrt{3} - 4\sqrt{2} + 1\right)}$$

Fubini's Theorem. If f is continuous on the rectangle $R = [a,b] \times [c,d]$, then

$$\boxed{\iint_R f(x,y)\, dA = \int_c^d \int_a^b f(x,y)dxdy = \int_a^b \int_c^d f(x,y)dydx}$$

Example 2. Calculate the double integral

$$\iint_R \left(xy^2 + \frac{y}{x}\right) dA,$$

where $R = \{(x,y)|2 \le x \le 3, -1 \le y \le 0\}$.

$$\iint_R \left(xy^2 + \frac{y}{x}\right) dA = \int_{-1}^{0}\left[\int_2^3 \left(xy^2+\frac{y}{x}\right)dx\right]dy = \int_2^3\int_{-1}^0 \left(xy^2+\frac{y}{x}\right) dy\,dx$$

$$= \int_{-1}^0 \left[\frac{x^2}{2}y^2 + y\ln|x|\right]_{x=2}^{x=3} dy = \int_{-1}^0 \left(\frac{9}{2}y^2 + y\ln 3 - 2y^2 - y\ln 2\right)dy$$

$$= \int_{-1}^0 \left(\frac{5y^2}{2} + y\ln\frac{3}{2}\right)dy = \left(\frac{5y^3}{6} + \frac{y^2}{2}\ln\frac{3}{2}\right)_{-1}^0 = -\left(-\frac{5}{6} + \frac{1}{2}\ln\frac{3}{2}\right)$$

$$= \boxed{\frac{5}{6} - \frac{1}{2} \ln\frac{3}{2}}$$

Example 3. Find the volume of the solid lying under the elliptic paraboloid $\frac{x^2}{4} + \frac{y^2}{9} + z = 1$ and above the rectangle $R = [-1,1] \times [-2,2]$.

$z = 1 - \frac{x^2}{4} - \frac{y^2}{9}$

$-1 \le x \le 1, \quad -2 \le y \le 2$

$$V = \int_{-1}^1\left(\int_{-2}^2 \left(1-\frac{x^2}{4}-\frac{y^2}{9}\right)dy\right)dx = \int_{-2}^2\int_{-1}^1 \left(1-\frac{x^2}{4}-\frac{y^2}{9}\right)dx\,dy$$

$$= \int_{-1}^1 \left(y - \frac{x^2}{4}y - \frac{y^3}{27}\right)_{y=-2}^{y=2} dx = \int_{-1}^1 \left(4 - 4\frac{x^2}{4} - \frac{16}{27}\right)dx$$

$$= \int_{-1}^1 \left(\frac{92}{27} - x^2\right)dx = \left(\frac{92}{27}x - \frac{x^3}{3}\right)_{-1}^1 = \frac{92}{27}(2) - \frac{2}{3} = \boxed{\frac{184}{27} - \frac{2}{3}}$$