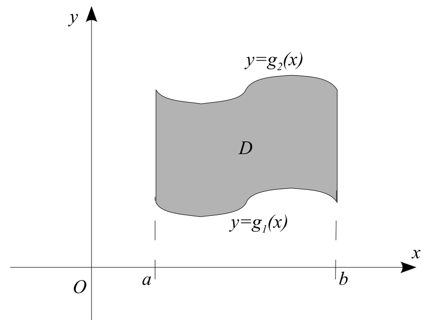
Section 15.2 **Double integrals over general regions.**

A plane region D is said to be of **type I** if it lies between the graphs of two continuous functions of f, that is,

$$D = \{(x, y)|x \le x \le b, g_1(x) \le y \le g_2(x)\}$$

where g_1 and g_2 are continuous on $[a, b]$.



In order to evaluate $\iint\limits_D f(x, y)dA$ when D is a region of type I, we choose a rectangle $R = [a, b] \times [c, d]$ that contains D and we let

$$F(x, y) = \begin{cases} f(x, y), & \text{if } (x, y) \in D \\ 0, & \text{if } (x, y) \text{ is in } R \text{ but not in } D \end{cases}$$

Then, by Fubini's Theorem,

$$\iint\limits_D f(x, y)dA = \iint\limits_R F(x, y)dA = \int\limits_a^b \int\limits_c^d F(x, y)\ dydx$$

Since $F(x, y) = 0$ if $y < g_1(x, y)$ or $y > g_2(x)$, then

$$\int\limits_c^d F(x, y)\ dy = \int\limits_c^{g_1(x)} F(x, y)\ dy + \int\limits_{g_1(x)}^{g_2(x)} F(x, y)\ dy + \int\limits_{g_2(x)}^{d} F(x, y)\ dy = \int\limits_{g_1(x)}^{g_2(x)} f(x, y)\ dy.$$

If f is continuous on a type I region D such that

$$D = \{(x, y)|x \le x \le b, g_1(x) \le y \le g_2(x)\}$$

then

$$\boxed{\iint\limits_D f(x, y)dA = \int\limits_a^b \int\limits_{g_1(x)}^{g_2(x)} f(x, y)\ dydx}$$

Example 1. Evaluate the integral

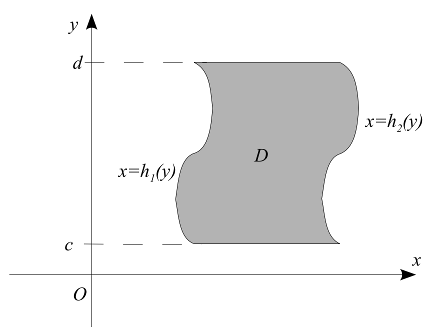
$$\iint_D xy \, dA$$

if $D = \{(x, y) | 0 \le x \le 1, x^2 \le y \le \sqrt{x}\}$.

We also consider the plane region of **type II**, which can be expressed as

$$D = \{(x, y) | c \le y \le d, h_1(y) \le x \le h_2(y)\}$$

where h_1 and h_2 are continuous.



We can show that

$$\boxed{\iint_D f(x, y) dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y) \, dx dy}$$

where D is a type II region.

Example 2. Evaluate the double integrals

$$\iint_D (y^2 - x)\ dA,$$

where D is bounded by $x = y^2$ and $x = 3 - 2y^2$.

Properties of double integrals.
We assume that all of the following integrals exist. Then

1. $\displaystyle\iint_D [f(x,y) + g(x,y)]dA = \iint_D f(x,y)dA + \iint_D g(x,y)dA$

2. $\displaystyle\iint_D cf(x,y)dA = c\iint_D f(x,y)dA$, where c is a constant

3. If $f(x,y) \geq g(x,y)$ for all (x,y) in D, then
$$\iint_D f(x,y)dA \geq \iint_R g(x,y)dA$$

4. If $D = D_1 \cup D_2$, where D_1 and D_2 do not overlap except perhaps on their boundaries, then
$$\iint_D f(x,y)dA = \iint_{D_1} f(x,y)dA + \iint_{D_2} f(x,y)dA$$

5. $\displaystyle\iint_D 1\ dA = A(D)$

6. If $m \leq f(x,y) \leq M$ for all (x,y) in D, then
$$mA(D) \leq \iint_D f(x,y)dA \leq MA(D).$$

The average value of a function.

The **average value** of a function f over a region D is

$$\boxed{f_{ave} = \frac{1}{A(D)} \iint\limits_D f(x,y)dA}$$

Example 3. Find the average value of the function $f(x,y) = ye^x$ over a triangular region with vertices $(0,0)$, $(2,4)$, and $(6,0)$.

Example 4. Sketch the region of integration and change the order of integration for

$$\int_0^1 \int_{y^2}^{2-y} f(x,y)\ dxdy$$

Example 5. Find the volume of the solid under the paraboloid $z = 3x^2 + y^2$ and above the region bounded by $y = x$ and $x = y^2 - y$.