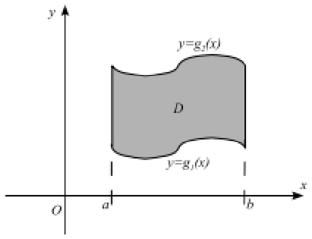
Section 15.2 **Double integrals over general regions.**

A plane region D is said to be of **type I** if it lies between the graphs of two continuous functions of f, that is,

$$D = \{(x,y) | x \le x \le b, g_1(x) \le y \le g_2(x)\}$$

where g_1 and g_2 are continuous on $[a,b]$.



In order to evaluate $\iint\limits_D f(x,y)dA$ when D is a region of type I, we choose a rectangle $R = [a,b] \times [c,d]$ that contains D and we let

$$F(x,y) = \begin{cases} f(x,y), & \text{if } (x,y) \in D \\ 0, & \text{if } (x,y) \text{ is in } R \text{ but not in } D \end{cases}$$

Then, by Fubini's Theorem,

$$\iint\limits_D f(x,y)dA = \iint\limits_R F(x,y)dA = \int_a^b \int_c^d F(x,y)\, dydx$$

Since $F(x,y) = 0$ if $y < g_1(x,y)$ or $y > g_2(x)$, then

$$\int_c^d F(x,y)\, dy = \int_c^{g_1(x)} F(x,y)\, dy + \int_{g_1(x)}^{g_2(x)} F(x,y)\, dy + \int_{g_2(x)}^{d} F(x,y)\, dy = \int_{g_1(x)}^{g_2(x)} f(x,y)\, dy.$$

If f is continuous on a type I region D such that

$$D = \{(x,y) | x \le x \le b, g_1(x) \le y \le g_2(x)\}$$

then

$$\boxed{\iint\limits_D f(x,y)dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\, dydx}$$

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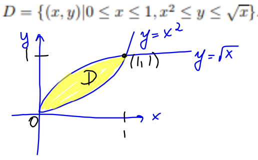
Example 1. Evaluate the integral

$$\iint\limits_D xy\, dA$$

if $D = \{(x,y) | 0 \le x \le 1, x^2 \le y \le \sqrt{x}\}$.



$$= \int_0^1 \left(\int_{x^2}^{\sqrt{x}} xy\, dy \right) dx$$

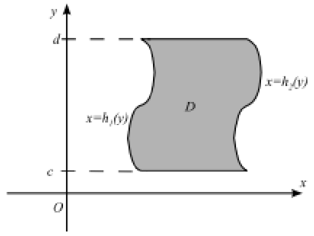
$$= \int_0^1 x \frac{y^2}{2} \Big|_{y=x^2}^{y=\sqrt{x}} dx = \frac{1}{2}\int_0^1 x(x - x^4)\, dx$$

$$= \frac{1}{2}\int_0^1 (x^2 - x^5)\, dx = \frac{1}{2}\left(\frac{x^3}{3} - \frac{x^6}{6}\right)\Big|_0^1 = \frac{1}{2}\left(\frac{1}{3} - \frac{1}{6}\right) = \boxed{\frac{1}{12}}$$

We also consider the plane region of **type II**, which can be expressed as

$$D = \{(x,y) | c \le y \le d, h_1(y) \le x \le h_2(y)\}$$

where h_1 and h_2 are continuous.



We can show that

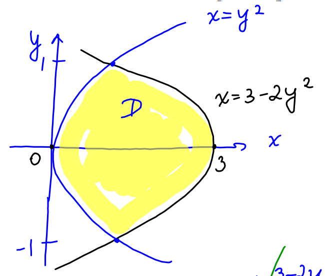
$$\boxed{\iint_D f(x,y)dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\ dxdy}$$

where D is a type II region.

Example 2. Evaluate the double ~~integrals~~

$$\iint_D (y^2 - x)\ dA,$$

where D is bounded by $x = y^2$ and $x = 3 - 2y^2$.



$y^2 \le x \le 3-2y^2$

Points of intersection:

$y^2 = 3-2y^2$

$3y^2 = 3$ or $y^2 = 1$ or $y = \pm 1$

$-1 \le y \le 1$

$$\iint_D (y^2-x)\,dA = \int_{-1}^{1}\left(\int_{y^2}^{3-2y^2}(y^2-x)\,dx\right)dy = \int_{-1}^{1}\left(y^2x-\frac{x^2}{2}\right)\Big|_{x=y^2}^{x=3-2y^2} dy$$

$$= \int_{-1}^{1}\left(y^2(3-2y^2-y^2) - \frac{1}{2}\left((3-2y^2)^2-y^4\right)\right)dy$$

$$= \int_{-1}^{1}\left(3y^2-3y^4-\frac{1}{2}(9-12y^2+4y^4-y^4)\right)dy = \int_{-1}^{1}\left(3y^2-3y^4-\frac{9}{2}+6y^2-\frac{3}{2}y^4\right)dy$$

$$= \int_{-1}^{1}\left(9y^2-\frac{9}{2}y^4-\frac{9}{2}\right)dy = \left(9\frac{y^3}{3}-\frac{9}{2}\frac{y^5}{5}-\frac{9}{2}y\right)\Big|_{-1}^{1}$$

$$= 6-\frac{9}{5}-9 = -\frac{9}{5}-3 = \boxed{-\frac{24}{5}}$$

Properties of double integrals.
We assume that all of the following integrals exist. Then

1. $\displaystyle\iint_D [f(x,y) \pm g(x,y)]dA = \iint_D f(x,y)dA \pm \iint_D g(x,y)dA$

2. $\displaystyle\iint_D cf(x,y)dA = c\iint_D f(x,y)dA$, where c is a constant

3. If $f(x,y) \geq g(x,y)$ for all (x,y) in D, then
$$\iint_D f(x,y)dA \geq \iint_R g(x,y)dA$$

4. If $D = D_1 \cup D_2$, where D_1 and D_2 do not overlap except perhaps on their boundaries, then
$$\iint_D f(x,y)dA = \iint_{D_1} f(x,y)dA + \iint_{D_2} f(x,y)dA$$

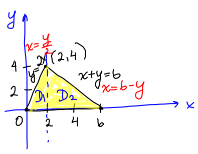
5. $\boxed{\displaystyle\iint_D 1\ dA = A(D)}$

6. If $m \leq f(x,y) \leq M$ for all (x,y) in D, then
$$mA(D) \leq \iint_D f(x,y)dA \leq MA(D).$$

The average value of a function.
The average value of a function f over a region D is

$$\boxed{f_{ave} = \frac{1}{A(D)}\iint_D f(x,y)dA}$$

Example 3. Find the average value of the function $f(x,y) = ye^x$ over a triangular region with vertices $(0,0)$ $(2,4)$, and $(6,0)$.



$A(\Delta) = \iint_\Delta 1\, dA =$

$\Delta = D_1 \cup D_2$

$D_1: \quad 0 \leq x \leq 2$
$\quad\ \ 0 \leq y \leq 2x$

$D_2: \quad 2 \leq x \leq 6$
$\quad\ \ 0 \leq y \leq 6-x$

$A = \iint_{D_1} 1\, dA + \iint_{D_2} 1 dA$

$0 \leq y \leq 4$
$\frac{y}{2} \leq x \leq 6-y$

$A(\Delta) = \int_0^4 \int_{\frac{y}{2}}^{6-y} 1\, dx\, dy = \int_0^4 x\Big|_{\frac{y}{2}}^{6-y} dy$

$= \int_0^4 \left(6 - y - \frac{y}{2}\right) dy$

$= \int_0^4 \left(6 - \frac{3}{2}y\right)dy = \left(6y - \frac{3y^2}{4}\right)\Big|_0^4$

$= 24 - 12 = 12$

$f_{ave} = \frac{1}{12}\iint_\Delta ye^x dA = \frac{1}{12}\int_0^4\left(\int_{\frac{y}{2}}^{6-y} ye^x dx\right)dy$

$= \frac{1}{12}\int_0^4 ye^x\Big|_{x=\frac{y}{2}}^{x=6-y} dy = \frac{1}{12}\int_0^4 \left(ye^{6-y} - ye^{\frac{y}{2}}\right) dy$

$= \frac{1}{12}\int_0^4 ye^{6-y} dy - \frac{1}{12}\int_0^4 ye^{\frac{y}{2}} dy$

$u = y \quad v' = e^{6-y} \quad | \quad u = y \quad v' = e^{\frac{y}{2}}$
$u' = 1 \quad v = -e^{6-y} \quad | \quad u' = 1 \quad v = 2e^{\frac{y}{2}}$

$= \frac{1}{12}\left(-ye^{6-y}\Big|_0^4 + \int_0^4 e^{6-y} dy\right) - \frac{1}{12}\left(2ye^{\frac{y}{2}}\Big|_0^4 - 2\int_0^4 e^{\frac{y}{2}}dy\right)$

$= \frac{1}{12}\left(-4e^2 - e^{6-y}\Big|_0^4\right) - \frac{1}{12}\left(8e^2 - 4e^{\frac{y}{2}}\Big|_0^4\right)$

$= \boxed{\frac{1}{12}\left(-4e^2 - e^2 + e^6\right) - \frac{1}{12}\left(8e^2 - 4e^2 + 4\right)}$

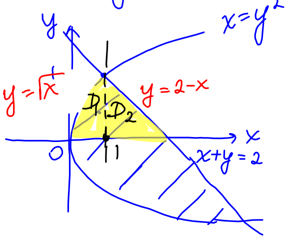
Example 4. Sketch the region of integration and change the order of integration for

$$\int_0^1 \int_{y^2}^{2-y} f(x,y)\,dx\,dy = \boxed{\int_0^1 \int_0^{\sqrt{x}} f(x,y)\,dy\,dx + \int_1^2 \int_0^{2-x} f(x,y)\,dy\,dx}$$

$x=y^2$, $x=2-y$ or $x+y=2$

$y^2 \le x \le 2-y$

$0 \le y \le 1$

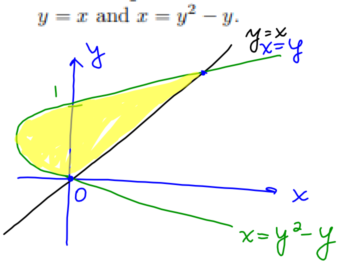


$D = D_1 \cup D_2$

$D_1:\ 0 \le x \le 1$, $0 \le y \le \sqrt{x}$

$D_2:\ 1 \le x \le 2$, $0 \le y \le 2-x$

Example 5. Find the volume of the solid under the paraboloid $z = 3x^2 + y^2$ and above the region bounded by $y = x$ and $x = y^2 - y$.



$y^2 - y \le x \le y$

$0 \le y \le 2$

Points of intersection:

$y^2 - y = y$

$y^2 - 2y = 0$

$y(y-2) = 0$

$y_1 = 0,\quad y_2 = 2$

$$\int_0^2 \int_{y^2-y}^{y} (3x^2+y^2)\,dx\,dy$$

$$= \int_0^2 \left[x^3 + y^2 x\right]_{x=y^2-y}^{x=y} dy$$

$$= \int_0^2 \left[y^3 + y^3 - (y^2-y)^3 - y^2(y^2-y)\right] dy$$

$(a-b)^3 = a^3 - 3a^2 b + 3ab^2 - b^3$

$$= \int_0^2 \left[2y^3 - (y^6 - 3y^5 + 3y^4 - y^3) - y^4 + y^3\right] dy$$

$$= \int_0^2 \left[4y^3 - y^6 + 3y^5 - 4y^4\right] dy = \ldots$$