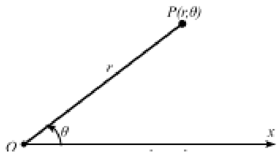
Section 15.3 **Double integrals in polar coordinates.**

We choose a point in the plane that is called the **pole** (or origin) and labeled O. Then we draw a ray (half-line) starting at O called the **polar axis**. This axis is usually drown horizontally to the right and corresponds to the positive x-axis in Cartesian coordinates.

If P is any point in the plane, let r be the distance from O to P and let θ be the angle (in radians) between the polar axis and the line OP. Then the point P is represented by the ordered pair (r, θ) and r, θ are called **polar coordinates** of P.

We use the convention that an angle is positive if measured in the counterclockwise direction from the polar axis and negative in the clockwise direction. If $P = 0$, then $r = 0$ and we agree that $(0, \theta)$ represents the pole for any value of θ.



In the Cartesian coordinate system every point has only one representation, but in the polar coordinate system each point has many representations. Since a complete counterclockwise rotation is given by an angle 2π, the point represented by polar coordinates (r, θ) is also represented by

$$(r, \theta + 2\pi n) \qquad \text{and} \qquad (-r, \theta + (2n+1)\pi),$$

where n is any integer.

The connection between polar and Cartesian coordinates is $\boxed{\begin{array}{l} x = r\cos\theta \\ y = r\sin\theta \end{array}}$ and $\boxed{\begin{array}{l} r^2 = x^2 + y^2 \\ \tan\theta = \dfrac{y}{x} \end{array}}$

Equation for θ do not uniquely determine it when x and y are given. Therefore, in converting from Cartesian to polar coordinates, it is not good enough just to find r and θ that satisfy equations. We must choose θ so that the point (r, θ) lies in correct quadrant.

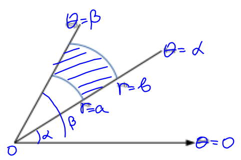
We want to evaluate

$$\iint_R f(x,y)\,dA,$$

where R is a **polar rectangle**

$$R = \{(r,\theta) | a \le r \le b, \alpha \le \theta \le \beta\}$$

$r = a$ - circle
$r^2 = x^2 + y^2$
$x^2 + y^2 = a^2$



Change to polar coordinates in a double integral. In polar coordinates

$$x = r\cos\theta \qquad dA = \left|\frac{\partial(x,y)}{\partial(r,\theta)}\right| dr d\theta$$
$$y = r\sin\theta$$

where $\dfrac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \dfrac{\partial x}{\partial r} & \dfrac{\partial x}{\partial \theta} \\ \dfrac{\partial y}{\partial r} & \dfrac{\partial y}{\partial \theta} \end{vmatrix}$ is the **Jacobian** of the transformation.

$x = r\cos\theta$
$\dfrac{\partial x}{\partial r} = \cos\theta \qquad \dfrac{\partial x}{\partial \theta} = -r\sin\theta$
$y = r\sin\theta$
$\dfrac{\partial y}{\partial r} = \sin\theta \qquad \dfrac{\partial y}{\partial \theta} = r\cos\theta$

Let us find the Jacobian.

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$

$$= r\cos^2\theta + r\sin^2\theta = r(\cos^2\theta + \sin^2\theta) = r$$

$$\boxed{dA = r\,dr\,d\theta}$$

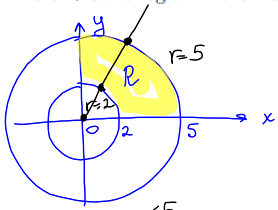
Thus, if f is continuous on a polar rectangle R given by $0 \le a \le r \le b$, $\alpha \le \theta \le \beta$, where $0 \le \alpha - \beta \le 2\pi$, then

$$\boxed{\iint_R f(x,y)\,dA = \int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta)\, r\,dr\,d\theta}$$

Example 1. Evaluate the integral

$$\iint_R xy\,dA$$

where R is the region in the first quadrant that lies between the circles $x^2 + y^2 = 4$ and $x^2 + y^2 = 25$.



Polar coordinates

$x = r\cos\theta$

$y = r\sin\theta$

$dA = r\,dr\,d\theta$

$x^2+y^2=4$

$r^2\cos^2\theta + r^2\sin^2\theta = 4$

$r^2(\cos^2\theta+\sin^2\theta)=4$

$r^2 = 4$

$r=2$

$x^2+y^2=25$

$r^2=25$

$r=5$

$2 \le r \le 5$

$0 \le \theta \le \frac{\pi}{2}$

$$\iint_R xy\,dA = \int_0^{\pi/2}\int_2^5 \underbrace{r\cos\theta}_{x}\,\underbrace{r\sin\theta}_{y}\,\underbrace{r\,dr\,d\theta}_{dA}$$

$$= \int_0^{\pi/2}\int_2^5 r^3\cos\theta\sin\theta\,dr\,d\theta = \int_0^{\pi/2} \frac{r^4}{4}\Big|_{r=2}^{r=5} \cos\theta\sin\theta\,d\theta$$

$$= \frac{609}{4}\int_0^{\pi/2}\cos\theta\sin\theta\,d\theta$$

$u = \sin\theta$

$du = \cos\theta\,d\theta$

$\theta = 0 \Rightarrow u = 0$

$\theta = \pi/2 \Rightarrow u = 1$

$$= \frac{609}{4}\int_0^1 u\,du = \frac{609}{4}\,\frac{u^2}{2}\Big|_0^1 = \boxed{\frac{609}{8}}$$

If f is continuous on a polar region of the form

$$D = \{(r,\theta) | \alpha \le \theta \le \beta, h_1(\theta) \le r \le h_2(\theta)\},$$

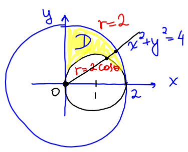
then

$$\boxed{\iint_D f(x,y)\,dA = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} f(r\cos\theta, r\sin\theta)\ r\,dr\,d\theta}$$

Example 2. Evaluate the integral

$$\iint_D x\,dA$$

where D is the region in the first quadrant that lies between the circles $x^2 + y^2 = 4$ and $x^2 + y^2 = 2x$.



$x = r\cos\theta$

$y = r\sin\theta$

$dA = r\,dr\,d\theta$

$x^2+y^2=4$

$r=2$

$x^2+y^2=2x$

$r^2 = 2r\cos\theta$

$r = 2\cos\theta$

$(x-1)^2+y^2=1$

$x^2-2x+y^2=0$

complete the square

$(x-1)^2+y^2=1$

$0 \le \theta \le \frac{\pi}{2}$

$2\cos\theta \le r \le 2$

$$\iint_D x\,dA = \int_0^{\pi/2}\int_{2\cos\theta}^{2} \underbrace{r\cos\theta}_{x}\,\underbrace{r\,dr\,d\theta}_{dA} = \int_0^{\pi/2} \frac{r^3}{3}\Big|_{r=2\cos\theta}^{2} \cos\theta\,d\theta$$

$$= \frac{8}{3}\int_0^{\pi/2}\left[1-\cos^3\theta\right]\cos\theta\,d\theta = \frac{8}{3}\int_0^{\pi/2}\cos\theta\,d\theta - \frac{8}{3}\int_0^{\pi/2}\cos^4\theta\,d\theta$$

$$\cos^2\theta = \frac{1+\cos 2\theta}{2}$$

$$\cos^4\theta = \left(\frac{1+\cos 2\theta}{2}\right)^2 = \frac{1}{4}\left(1+2\cos 2\theta+\cos^2 2\theta\right) = \frac{1}{4} + \frac{1}{2}\cos 2\theta + \frac{1}{4}\,\frac{1+\cos 4\theta}{2}$$

$$\cos^4\theta = \frac{3}{8} + \frac{1}{2}\cos 2\theta + \frac{1}{8}\cos 4\theta$$

$$= \frac{8}{3}\sin\theta\Big|_0^{\pi/2} - \frac{8}{3}\int_0^{\pi/2}\left(\frac{3}{8} + \frac{1}{2}\cos 2\theta + \frac{1}{8}\cos 4\theta\right)d\theta$$

$$= \frac{8}{3} - \frac{8}{3}\left(\frac{3}{8}\theta + \frac{1}{4}\sin 2\theta + \frac{1}{32}\sin 4\theta\right)\Big|_0^{\pi/2}$$

$$= \boxed{\frac{8}{3} - \frac{\pi}{2}}$$

Example 4. Use a double integral to find the area of the region inside the circle $r = 3\cos\theta$ and outside the cardioid $r = 1 + \cos\theta$.

$r^2 = 3r\cos\theta$

$x^2 + y^2 = 3x$

$\left(x - \frac{3}{2}\right)^2 + y^2 = \frac{9}{4}$

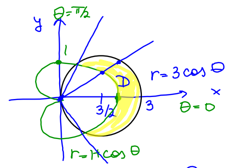
$r = 1 + \cos\theta$

$\theta = 0 \Rightarrow r = 2$

$\theta = \frac{\pi}{2} \Rightarrow r = 1$

$\theta = \frac{3\pi}{4} \Rightarrow r = 1 - \frac{\sqrt{2}}{2}$

$\theta = \pi \Rightarrow r = 0$



$1 + \cos\theta \le r \le 3\cos\theta$

$-\frac{\pi}{6} \le \theta \le \frac{\pi}{6}$

Points of intersection:

$1 + \cos\theta = 3\cos\theta$

$2\cos\theta = 1$

$\cos\theta = \frac{1}{2}$

$\theta = \frac{\pi}{3}, \quad \theta = -\frac{\pi}{3}$

$$A = \iint_D 1\, dA \quad \left| \begin{array}{l} x = r\cos\theta \\ y = r\sin\theta \\ dA = r\,dr\,d\theta \end{array} \right| = \int_{-\pi/3}^{\pi/3} \int_{1+\cos\theta}^{3\cos\theta} r\,dr\,d\theta = 2\int_0^{\pi/3} \int_{1+\cos\theta}^{3\cos\theta} r\,dr\,d\theta$$

$$= \cancel{2}\int_0^{\pi/3} \frac{r^2}{\cancel{2}}\Big|_{1+\cos\theta}^{3\cos\theta} d\theta = \int_0^{\pi/3} \Big[9\cos^2\theta - \overbrace{(1+\cos\theta)^2}^{1+2\cos\theta+\cos^2\theta}\Big] d\theta$$

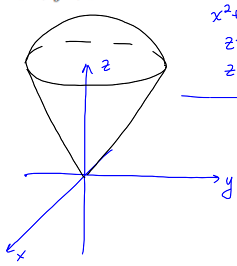
$$= \int_0^{\pi/3} \left[9\cos^2\theta - 1 - 2\cos\theta - \cos^2\theta\right] d\theta$$

$$= \int_0^{\pi/3} \left(8\cos^2\theta - 1 - 2\cos\theta\right) d\theta = \int_0^{\pi/3} \left(8 \cdot \frac{1+\cos 2\theta}{2} - 1 - 2\cos\theta\right) d\theta$$

$$= \int_0^{\pi/3} \left(4 + 4\cos 2\theta - 1 - 2\cos\theta\right) d\theta = \int_0^{\pi/3} \left(3 + 4\cos 2\theta - 2\cos\theta\right) d\theta$$

$$= \left(3\theta + \frac{4}{2}\sin 2\theta - 2\sin\theta\right)\Big|_0^{\pi/3} = 3 \cdot \frac{\pi}{3} + 2\sin\frac{2\pi}{3} - 2\sin\frac{\pi}{3} = \boxed{\pi}$$

Example 5. Use polar coordinates to find the volume above the cone $z = \sqrt{x^2 + y^2}$ and below the sphere $x^2 + y^2 + z^2 = 1$.



$x^2 + y^2 + z^2 = 1$

$z^2 = 1 - x^2 - y^2$

$z = \sqrt{1 - x^2 - y^2}$

$$V = \iint_D \left(\sqrt{1-x^2-y^2} - \sqrt{x^2+y^2}\right) dA \quad \left| \begin{array}{l} x = r\cos\theta \\ y = r\sin\theta \\ x^2+y^2 = r^2 \\ dA = r\,dr\,d\theta \end{array} \right|$$

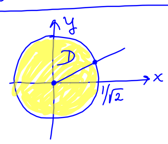
Curve of intersection:

$\begin{cases} z = \sqrt{x^2+y^2} \\ x^2 + y^2 + z^2 = 1 \end{cases}$

$x^2 + y^2 + (x^2 + y^2) = 1$

$2x^2 + 2y^2 = 1 \quad \text{or} \quad x^2 + y^2 = \frac{1}{2}$

Projection onto the (xy)-plane:



$x^2 + y^2 = 1/2$

$r = 1/\sqrt{2}$

$0 \le r \le \frac{1}{\sqrt{2}}$

$0 \le \theta \le 2\pi$

$$V = \int_0^{2\pi} \int_0^{\frac{1}{\sqrt{2}}} \left(\sqrt{1-r^2} - r\right) r\,dr\,d\theta$$

$$= \int_0^{2\pi} \int_0^{\frac{1}{\sqrt{2}}} r\sqrt{1-r^2}\,dr\,d\theta - \int_0^{2\pi} \int_0^{\frac{1}{\sqrt{2}}} r^2\,dr\,d\theta$$

$u = 1 - r^2$

$du = -2r\,dr$

$\left| \begin{array}{l} r = 0 \Rightarrow u = 1 \\ r = \frac{1}{\sqrt{2}} \Rightarrow u = 1 - \frac{1}{2} = \frac{1}{2} \end{array} \right.$

$$= -\frac{1}{2}\int_0^{2\pi} \int_1^{1/2} \sqrt{u}\,du\,d\theta - \int_0^{2\pi} \frac{r^3}{3}\Big|_0^{1/\sqrt{2}} d\theta$$

$$= -\frac{1}{2} \int^{2\pi} \frac{u^{3/2}}{3/2}\Big|_1^{1/2} d\theta - \frac{1}{3}\left(\frac{1}{2\sqrt{2}}\right) \int d\theta$$

$$= -\frac{1}{2} \cdot \frac{2}{3}\left(\frac{1}{2\sqrt{2}} - 1\right) \int_0^{2\pi} d\theta - \frac{1}{6\sqrt{2}}\, \theta\Big|_0^{2\pi}$$

$$= -\frac{1}{3}\left(\frac{1}{2\sqrt{2}} - 1\right) \theta\Big|_0^{2\pi} - \frac{1}{6\sqrt{2}}(2\pi) = 2\pi\left[\frac{1}{3} - \frac{1}{3\sqrt{2}}\right] \quad \boxed{\frac{2\pi}{3}\left(1 - \frac{1}{\sqrt{2}}\right)}$$