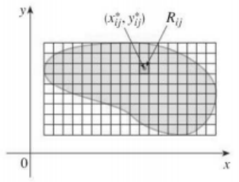
Section 15.4 **Applications of double integrals.**

Suppose the lamina occupies the region D on the xy-plane and its density (in units of mass per unit area) at a point (x, y) in D is given by $\rho(x, y)$, where ρ is a continuous function on D.



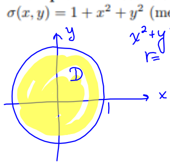
To find the total mass m of the lamina we partition a rectangle R containing D into subrectangles R_{ij} and consider $\rho(x, y)$ be zero outside D. If we choose a point $(x_{ij}^*, y_{ij}^*) \in R_{ij}$, then the mass of the part of the lamina that occupies R_{ij} is approximately $\rho(x_{ij}^*, y_{ij}^*)\Delta A_{ij}$, where ΔA_{ij} is the area of R_{ij}. If we add all such masses, we get an approximation to the total mass:

$$m \approx \sum_{i=1}^{m}\sum_{j=1}^{n} \rho(x_{ij}^*, y_{ij}^*)\Delta A_{ij} \underset{\|P\|\to 0}{\longrightarrow} \boxed{\iint_D \rho(x, y)dA = m}$$

If an electric charge is distributed over a region D and the charge density (in units of charge per unit area) is given by $\sigma(x, y)$ at a point (x, y) in D, then the total charge Q is given by

$$\boxed{Q = \iint_D \sigma(x, y)dA}$$

Example 1. Electric charge is distributed over the the unit disk $x^2 + y^2 \le 1$ so that the charge density at (x, y) is $\sigma(x, y) = 1 + x^2 + y^2$ (measured in coulombs per square meter). Find the total charge on the disk.



$x^2+y^2=1$
$r=1$
$0 \le r \le 1$
$0 \le \theta \le 2\pi$

Total charge

$$Q = \iint_D \sigma(x,y)\,dA = \iint_D (1+x^2+y^2)\,dA$$

$x = r\cos\theta$
$y = r\sin\theta$
$x^2+y^2=r^2$
$dA = r\,dr\,d\theta$

$$= \int_0^{2\pi}\int_0^1 (1+r^2)\,r\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^1 (r+r^3)\,dr\,d\theta = \int_0^{2\pi}\left(\frac{r^2}{2}+\frac{r^4}{4}\right)\Big|_0^1 d\theta$$

$$= \frac{3}{4}\int_0^{2\pi} d\theta = \frac{3}{4}\,\theta\Big|_0^{2\pi} = \boxed{\frac{3\pi}{2}}$$

Now let's find the center of mass of a lamina with density function $\rho(x, y)$ that occupies a region D. We define the moment of a particle about an axis as the product of its mass and its directed distance from the axis. If we partition D into small subrectangles, then the mass of R_{ij} is approximately $\rho(x_i^*, y_i^*)\Delta A_{ij}$, so we can approximate the moment of R_{ij} with respect to the x-axis by

$$[\rho(x_i^*, y_i^*)\Delta A_{ij}]y_{ij}^*$$

Thus, the moment of the entire lamina about the x-axis is

$$M_x = \lim_{\|P\|\to 0}\sum_{i=1}^{m}\sum_{j=1}^{n} y_{ij}^*\rho(x_i^*, y_i^*)\Delta A_{ij} = \boxed{\iint_D y\rho(x, y)dA = M_x}$$

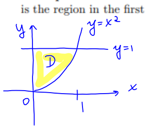
Similarly, the moment about the y-axis is

$$M_y = \lim_{\|P\|\to 0}\sum_{i=1}^{m}\sum_{j=1}^{n} x_{ij}^*\rho(x_i^*, y_i^*)\Delta A_{ij} = \boxed{\iint_D x\rho(x, y)dA = M_y}$$

We define the center of mass $(\bar{x}, \bar{y})$ so that $m\bar{x} = M_y$ and $m\bar{y} = M_x$. The physical significance is that the lamina behaves as if its entire mass is concentrated at its center of mass. Thus, the lamina balances horizontally when supported at its center of mass.

$$\boxed{\bar{x} = \frac{M_y}{m} = \frac{\iint_D x\rho(x, y)dA}{\iint_D \rho(x, y)dA}} \qquad \boxed{\bar{y} = \frac{M_x}{m} = \frac{\iint_D y\rho(x, y)dA}{\iint_D \rho(x, y)dA}}$$

Example 2. Find the center of mass of the lamina with density $\rho(x, y) = xy$ that occupies the region D, where D is the region in the first quadrant bounded by the parabola $y = x^2$ and the line $y = 1$.



$x^2 \le y \le 1$
$0 \le x \le 1$

Total mass:

$$m = \iint_D \rho(x,y)\,dA = \int_0^1\int_{x^2}^1 xy\,dy\,dx$$

$$= \int_0^1 x\,\frac{y^2}{2}\Big|_{x^2}^1 dx = \frac{1}{2}\int_0^1 x(1-x^4)\,dx$$

$$= \frac{1}{2}\int_0^1 (x-x^5)\,dx = \frac{1}{2}\left(\frac{x^2}{2}-\frac{x^6}{6}\right)\Big|_0^1 = \frac{1}{2}\left(\frac{1}{2}-\frac{1}{6}\right) = \boxed{\frac{1}{6}}$$

Moment about the x-axis

$$M_x = \iint_D y\,\rho(x,y)\,dA = \int_0^1\int_{x^2}^1 xy^2\,dy\,dx = \int_0^1 x\,\frac{y^3}{3}\Big|_{x^2}^1 dx = \frac{1}{3}\int_0^1 x(1-x^6)\,dx$$

$$= \frac{1}{3}\int_0^1 (x-x^7)\,dx = \frac{1}{3}\left(\frac{x^2}{2}-\frac{x^8}{8}\right)\Big|_0^1 = \frac{1}{3}\left(\frac{1}{2}-\frac{1}{8}\right) = \boxed{\frac{1}{8}}$$

Moment about the y-axis:

$$M_y = \iint_D x\rho(x,y)\,dA = \int_0^1\int_{x^2}^1 x^2y\,dy\,dx = \frac{1}{2}\int_0^1 x^2y^2\Big|_{x^2}^1 dx = \frac{1}{2}\int_0^1 x^2(1-x^4)\,dx$$

$$= \frac{1}{2}\int_0^1 (x^2-x^6)\,dx = \frac{1}{2}\left(\frac{x^3}{3}-\frac{x^7}{7}\right)\Big|_0^1 = \frac{1}{2}\left(\frac{1}{3}-\frac{1}{7}\right) = \boxed{\frac{2}{21}}$$

Center of mass: $(\bar{x}, \bar{y}) = \left(\frac{M_y}{m}, \frac{M_x}{m}\right) = \left(\frac{\frac{2}{21}}{\frac{1}{6}}, \frac{\frac{1}{8}}{1/6}\right) = \boxed{\left(\frac{4}{7}, \frac{3}{4}\right)}$