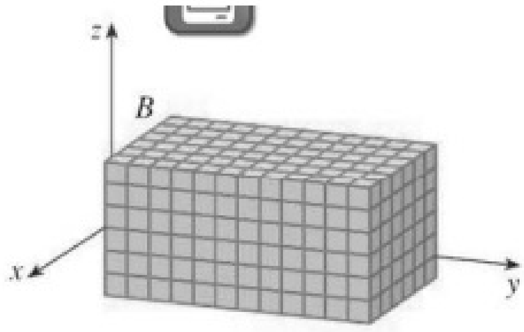
Section 15.6 **Triple integrals.**

We want to define the triple integrals for functions of three variables.
Let f is defined on a rectangular box

$$B = \{(x, y, z) | a \leq x \leq b, c \leq y \leq d, r \leq z \leq s\} = [a, b] \times [c, d] \times [r, s]$$



We partition the intervals $[a,b]$, $[c,d]$, and $[r,s]$ as follows:

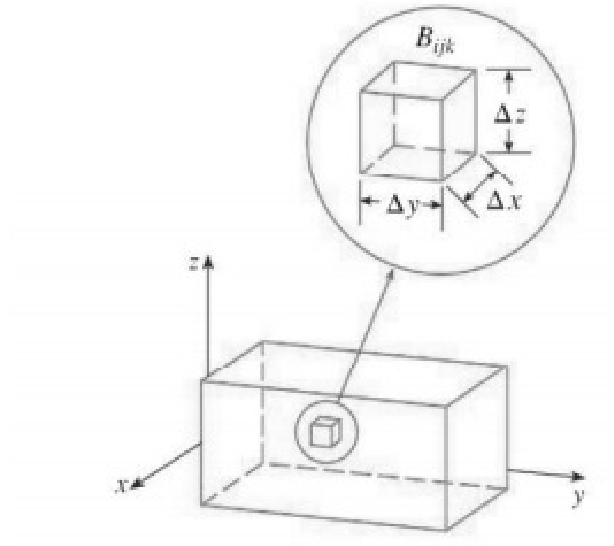
$$a = x_0 < x_1 < \ldots < x_m = b$$

$$c = y_0 < y_1 < \ldots < y_n = d$$

$$r = z_0 < z_1 < \ldots < z_k = s$$

The planes through these partition points parallel to coordinate planes divide the box B into lmn sub-boxes

$$B_{ijk} = [x_{i-1}, x_i] \times [y_{j-1}, y_j] \times [z_{k-1}, z_k]$$



The volume of B_{ijk} is

$$\Delta V_{ijk} = \Delta x_i \Delta y_j \Delta z_k$$

where $\Delta x_i = x_i - x_{i-1}$, $\Delta y_j = y_j - y_{j-1}$, and $\Delta z_k = z_k - z_{k-1}$.

Then we form the **triple Riemann sum**

$$\sum_{i=1}^{l}\sum_{j=1}^{m}\sum_{k=1}^{n} f(x_{ijk}^*, y_{ijk}^*, z_{ijk}^*)\Delta V_{ijk}$$

where $(x_{ijk}^*, y_{ijk}^*, z_{ijk}^*) \in B_{ijk}$. We define the **norm** $\|P\|$ of the partition P to be the length of the longest diagonal of all the boxes B_{ijk}.

Definition. The **triple integral** of f over the box B is

$$\iiint_B f(x,y,z)dV = \lim_{\|P\|\to 0}\sum_{i=1}^{l}\sum_{j=1}^{m}\sum_{k=1}^{n} f(x_{ijk}^*, y_{ijk}^*, z_{ijk}^*)\Delta V_{ijk}$$

if this limit exists.

Fubini's Theorem for triple integrals. If f is continuous on the rectangular box $B = [a,b] \times [c,d] \times [r,s]$, then

$$\boxed{\iiint_B f(x,y,z)dV = \int_r^s \int_c^d \int_a^b f(x,y,z)dxdydz}$$

There are five other possible orders in which we can integrate.

Example 1. Evaluate the integral $\iiint_E (x^2 + yz)dV$, where

$$E = \{(x,y,z) | 0 \le x \le 2,\ -3 \le y \le 0,\ -1 \le z \le 1\}$$

$$\iiint_E (x^2+yz)\,dV = \int_{-1}^{1}\int_{-3}^{0}\int_{0}^{2} (x^2+yz)\,dx\,dy\,dz = \int_{-1}^{1}\int_{-3}^{0} \left(\frac{x^3}{3} + xyz\right)_{x=0}^{x=2} dy\,dz$$

$$= \int_{-1}^{1}\int_{-3}^{0} \left(\frac{8}{3} + 2yz\right)dy\,dz = \int_{-1}^{1} \left(\frac{8}{3}y + y^2 z\right)_{y=-3}^{y=0} dz$$

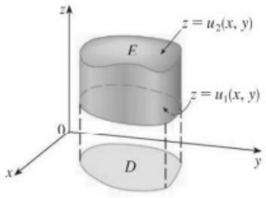
$$= \int_{-1}^{1} (8 - 9z)\,dz = \left(8z - \frac{9z^2}{2}\right)_{-1}^{1} = 8 + 8 = \boxed{16}$$

Now we define the triple integral over a general bounded region E in three-dimensional space.

A solid region E is said to be of type 1 if it lies between the graphs of two continuous functions of x and y, that is,

$$E = \{(x, y, z) | (x, y) \in D, u_1(x, y) \le z \le u_2(x, y)\}$$

where D is the projection of E onto the xy-plane.



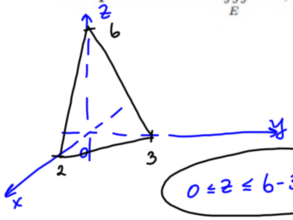
Then

$$\boxed{\iiint_E f(x, y, z)\,dV = \iint_D \left[\int_{u_1(x,y)}^{u_2(x,y)} f(x, y, z)\,dz\right] dA}$$

Example 2. Evaluate $\iiint_E x\,dV$, where E is bounded by planes $x = 0$, $y = 0$, $z = 0$, and $\frac{3x + 2y + z}{6} = \frac{6}{6}$. $\Rightarrow z = 6 - 3x - 2y$

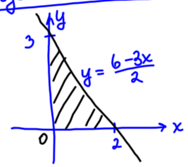
$\frac{x}{2} + \frac{y}{3} + \frac{z}{6} = 1$

$(2, 0, 0), (0, 3, 0), (0, 0, 6)$



$0 \le z \le 6 - 3x - 2y$

Projection onto the (xy)-plane



$z = 6 - 3x - 2y$

$z = 0$

$6 - 3x - 2y = 0$

$3x + 2y = 6$

$0 \le y \le \frac{6-3x}{2}$

$0 \le x \le 2$

$$\int_0^2 \int_0^{\frac{6-3x}{2}} \int_0^{6-3x-2y} x\,dz\,dy\,dx$$

$$= \int_0^2 \int_0^{\frac{6-3x}{2}} xz \Big|_{z=0}^{z=6-3x-2y} dy\,dx$$

$$= \int_0^2 \int_0^{\frac{6-3x}{2}} x(6 - 3x - 2y)\,dy\,dx$$

$$= \int_0^2 \int_0^{\frac{6-3x}{2}} (6x - 3x^2 - 2xy)\,dy\,dx$$

$$= \int_0^2 (6xy - 3x^2y - xy^2)\Big|_{y=0}^{y=\frac{6-3x}{2}} dx = \int_0^2 \left[3x(6-3x) - 3x^2\,\frac{6-3x}{2} - x\cdot\left(\frac{6-3x}{2}\right)^2\right] dx$$

$$= \int_0^2 \left[18x - 9x^2 - 9x^2 + \frac{9x^3}{2} - x\left(\frac{36 - 36x + 9x^2}{4}\right)\right] dx$$

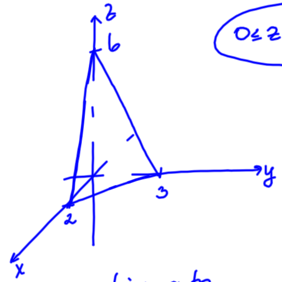
$$= \int_0^2 \left[18x - 18x^2 + \frac{9}{2}x^3 - 9x + 9x^2 - \frac{9x^3}{4}\right] dx = \int_0^2 \left(9x - 9x^2 + \frac{9}{4}x^3\right) dx$$

$$= 9\int_0^2 \left(x - x^2 + \frac{x^3}{4}\right) dx = 9\left(\frac{x^2}{2} - \frac{x^3}{3} + \frac{x^4}{16}\right)\Bigg|_{x=0}^{x=2} = 9\left(2 - \frac{8}{3} + 1\right) = \boxed{3}$$

$$\frac{\quad}{6} \quad \frac{\quad}{6} \Rightarrow z = 6-3x-2y$$

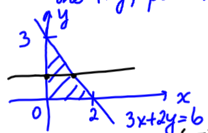
$$\frac{x}{2} + \frac{y}{3} + \frac{z}{6} = 1$$

$$(2,0,0),\ (0,3,0),\ (0,0,6)$$



$$0 \le z \le 6-3x-2y$$

Projection onto the (xy)-plane.



$$3x+2y=6$$

$$x = \frac{6-2y}{3}$$

$$0 \le y \le 3$$

$$0 \le x \le \frac{6-2y}{3}$$

$$\iiint_E x\,dV = \int_0^3 \int_0^{\frac{6-2y}{3}} \int_0^{6-3x-2y} x\,dz\,dx\,dy$$

$$= \int_0^3 \int_0^{\frac{6-2y}{3}} xz \Big|_{z=0}^{z=6-3x-2y} dx\,dy$$

$$= \int_0^3 \int_0^{\frac{6-2y}{3}} x(6-3x-2y)\,dx\,dy = \int_0^3 \left(\int_0^{\frac{6-2y}{3}} (6x-3x^2-2xy)\,dx \right) dy$$

$$= \int_0^3 \left(3x^2 - x^3 - x^2 y\right)_{x=0}^{x=\frac{6-2y}{3}} dy$$

$$= \int_0^3 \left[3\left(\frac{6-2y}{3}\right)^2 - \left(\frac{6-2y}{3}\right)^3 - \left(\frac{6-2y}{3}\right)^2 y \right] dy$$

$$= \int_0^3 \left(\frac{6-2y}{3}\right)^2 \left[3 - \frac{6-2y}{3} - y \right] dy = \int_0^3 \frac{36-24y+4y^2}{9} \cdot \frac{9-(6-2y)-3y}{3}\, dy$$

$$= \frac{1}{27} \int_0^3 (36-24y+4y^2)(3-y)\,dy = \frac{1}{27} \int_0^3 \left(108 - 72y + 12y^2 - 36y + 24y^2 - 4y^3\right) dy$$

$$= \frac{1}{27} \int_0^3 \left(108 - 108y + 36y^2 - 4y^3\right) dy = \frac{1}{27} \left(108y - 108\frac{y^2}{2} + 36\frac{y^3}{3} - \frac{4y^4}{4}\right)_0^3$$

$$= \boxed{\frac{1}{27}\left(108(3) - 108\cdot\frac{9}{2} + 36(3) - 81\right)}$$

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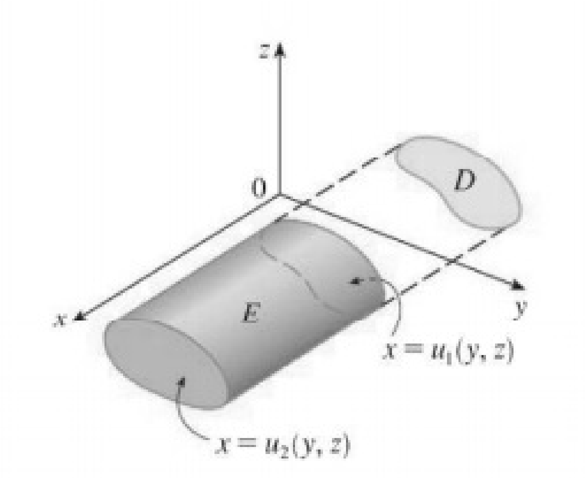
A solid region E is of **type 2** if it is of the form

$$E = \{(x, y, z) | (y, z) \in D, u_1(y, z) \leq x \leq u_2(y, z)\}$$

where D is the projection of E onto the yz-plane.



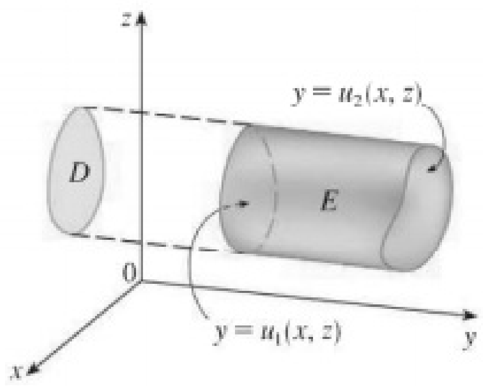
Then

$$\boxed{\iiint\limits_E f(x, y, z) dV = \iint\limits_D \left[\int_{u_1(y,z)}^{u_2(y,z)} f(x, y, z) dx \right] dA}$$

A solid region E is of **type 3** if it is of the form

$$E = \{(x, y, z) | (x, z) \in D, u_1(x, z) \le y \le u_2(x, z)\}$$

where D is the projection of E onto the xz-plane.



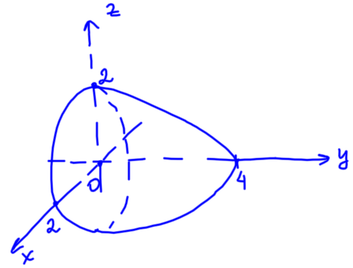
Then

$$\boxed{\iiint_E f(x, y, z) dV = \iint_D \left[\int_{u_1(x,z)}^{u_2(x,z)} f(x, y, z) dy \right] dA}$$

Example 3. Express the integral $\iiint_E f(x,y,z)\,dV$ as an iterated integral in six different ways, where E is the solid bounded by

$$y = 4 - x^2 - z^2, \qquad y = 0$$

xz-plane



$0 \le y \le 4 - x^2 - z^2$

$x^2 = 4 - y - z^2$

$x = \pm\sqrt{4-y-z^2}$

$-\sqrt{4-y-z^2} \le x \le \sqrt{4-y-z^2}$

$z^2 = 4 - x^2 - y$

$z = \pm\sqrt{4-x^2-y}$

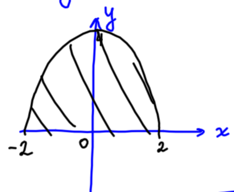
$-\sqrt{4-x^2-y} \le z \le \sqrt{4-x^2-y}$

Projections:

(xy)-plane

$z=0$

$y = 4 - x^2$



$x^2 = 4 - y$

$x = \pm\sqrt{4-y}$

$-\sqrt{4-y} \le x \le \sqrt{4-y}$

$0 \le y \le 4$

$0 \le y \le 4 - x^2$

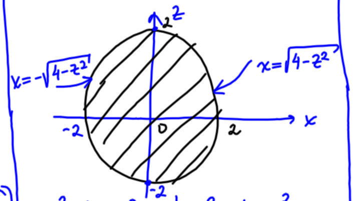
$-2 \le x \le 2$

(xz)-plane

$y=0$

$0 = 4 - x^2 - z^2$

$x^2 + z^2 = 4$



$x^2 = 4 - z^2$

$x = \pm\sqrt{4-z^2}$

$-\sqrt{4-z^2} \le x \le \sqrt{4-z^2}$

$-2 \le z \le 2$

$z^2 = 4 - x^2$

$z = \pm\sqrt{4-x^2}$

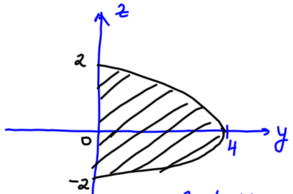
$-\sqrt{4-x^2} \le z \le \sqrt{4-x^2}$

$-2 \le x \le 2$

(yz)-plane

$x=0$

$y = 4 - z^2$



$0 \le y \le 4 - z^2$

$-2 \le z \le 2$

$z^2 = 4 - y$

$z = \pm\sqrt{4-y}$

$-\sqrt{4-y} \le z \le \sqrt{4-y}$

$0 \le y \le 4$

$$\iiint_E f(x,y,z)\,dV = \int_{-2}^{2}\int_{-\sqrt{4-z^2}}^{\sqrt{4-z^2}}\int_{0}^{4-x^2-z^2} f\,dy\,dx\,dz = \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{0}^{4-x^2-z^2} f\,dy\,dz\,dx$$

$$= \int_{-2}^{2}\int_{0}^{4-z^2}\int_{-\sqrt{4-y-z^2}}^{\sqrt{4-y-z^2}} f\,dx\,dy\,dz = \int_{0}^{4}\int_{-\sqrt{4-y}}^{\sqrt{4-y}}\int_{-\sqrt{4-y-z^2}}^{\sqrt{4-y-z^2}} f\,dx\,dz\,dy$$

$$= \int_{0}^{4}\int_{-\sqrt{4-y}}^{\sqrt{4-y}}\int_{-\sqrt{4-x^2-y}}^{\sqrt{4-x^2-y}} f\,dz\,dx\,dy = \int_{-2}^{2}\int_{0}^{4-x^2}\int_{-\sqrt{4-x^2-y}}^{\sqrt{4-x^2-y}} f\,dz\,dy\,dx$$

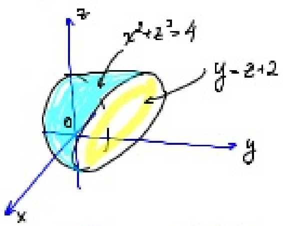
Applications of triple integrals.

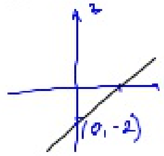
$$\boxed{V(E) = \iiint_E dV}$$

Example 4. Find the volume of the solid bounded by the elliptic cylinder $x^2 + z^2 = 4$ and the planes $y = 0$ and $y = z + 2$.

Example 4. Find the volume of the solid bounded by the elliptic cylinder $x^2 + z^2 = 4$ and the planes $y = 0$ and $y = z + 2$.

xz-plane





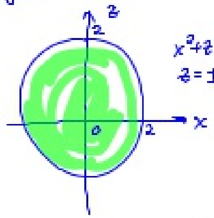
$y = z+2$

$y - z = 2$

$\frac{y}{2} - \frac{z}{2} = 1$

$0 \le y \le z+2$

Projection onto the (xz)-plane



$x^2+z^2=4$

$z = \pm\sqrt{4-x^2}$

$-\sqrt{4-x^2} \le z \le \sqrt{4-x^2}$

$-2 \le x \le 2$

Polar coordinates:

$x = r\cos\theta$

$z = r\sin\theta$

$0 \le r \le 2$

$0 \le \theta \le 2\pi$

$dx\,dz = r\,dr\,d\theta$

$V = \iiint_E 1\,dV$

$dV = dx\,dy\,dz = dy\,(\underbrace{r\,dr\,d\theta}_{dx\,dz})$

$dV = r\,dy\,dr\,d\theta$

$0 \le y \le z+2$

$0 \le y \le r\sin\theta + 2$

$0 \le r \le 2$

$0 \le \theta \le 2\pi$

$$V = \int_0^{2\pi}\int_0^{2}\int_0^{r\sin\theta+2} 1\cdot r\,dy\,dr\,d\theta = \int_0^{2\pi}\int_0^{2} ry\Big|_{y=0}^{y=r\sin\theta+2} dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^{2} r(r\sin\theta+2)\,dr\,d\theta = \int_0^{2\pi}\int_0^{2}(r^2\sin\theta+2r)\,dr\,d\theta$$

$$= \int_0^{2\pi}\left(\frac{r^3}{3}\sin\theta + r^2\right)\Big|_{r=0}^{r=2} d\theta = \int_0^{2\pi}\left(\frac{8}{3}\sin\theta + 4\right)d\theta$$

$$= \left(-\frac{8}{3}\cos\theta + 4\theta\right)\Big|_0^{2\pi} = \boxed{8\pi}$$