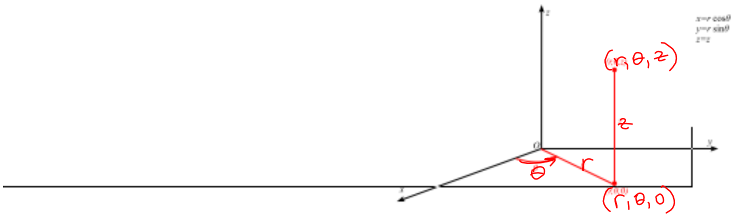
Section 15.7 **Triple integrals in cylindrical coordinates.**

Cylindrical coordinate system:



$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases}$$

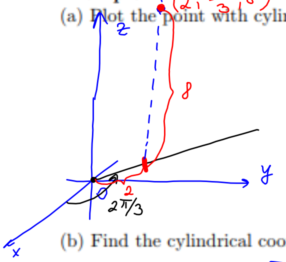
$dV = r\,dz\,dr\,d\theta$

To convert from rectangular to cylindrical coordinates we use

$$\boxed{r^2 = x^2 + y^2 \qquad \tan\theta = \frac{y}{x} \qquad z = z}$$

Example 1. $(2, \frac{2\pi}{3}, 8)$

(a) Plot the point with cylindrical coordinates $(2, 2\pi/3, 8)$ and find its rectangular coordinates.



$r = 2$ | $x = r\cos\theta = 2\cos\frac{2\pi}{3} = -1$

$\theta = \frac{2\pi}{3}$ | $y = r\sin\theta = 2\sin\frac{2\pi}{3} = \sqrt{3}$

$z = 8$ | $z = 8$

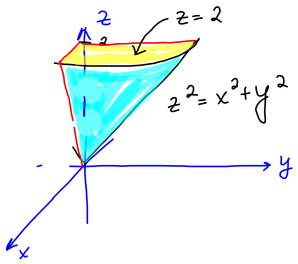
$\boxed{(-1, \sqrt{3}, 8)}$ cartesian coordinates.

(b) Find the cylindrical coordinates of the point with rectangular coordinates $(-\sqrt{2}, \sqrt{2}, 0)$.

$x = -\sqrt{2}$, $y = \sqrt{2}$, $z = 0$. | $r = 2$

$r^2 = x^2 + y^2 = 4$ | $\tan\theta = \frac{y}{x} = -1 \Rightarrow \theta = \frac{3\pi}{4}$ | $\boxed{(2, \frac{3\pi}{4}, 0)}$ cylindrical coordinates

Example 2. Sketch the solid given by the inequalities



$$0 \le \theta \le \pi/2, \qquad r \le z \le 2$$

$z = r$, $r^2 = x^2 + y^2$

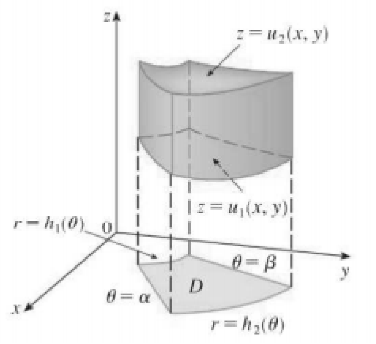
$z = \sqrt{x^2 + y^2}$

$z^2 = x^2 + y^2$ top half of the cone.

Suppose that E is a type 1 region whose projection D on the xy-plane is described in polar coordinates

$$E = \{(x, y, z) | (x, y) \in D, u_1(x, y) \leq z \leq u_2(x, y)\}$$

$$D = \{(r, \theta) | \alpha \leq \theta \leq \beta, h_1(\theta) \leq r \leq h_2(\theta)\}$$



Then

$$\iiint_E f(x, y, z) dV = \iint_D \left[\int_{\varphi_1(x,y)}^{\varphi_2(x,y)} f(x, y, z) dz \right] dA$$

If we switch to cylindrical coordinates

$$\boxed{x = r\cos\theta, \quad y = r\sin\theta, \quad z = z, \quad dA = r\,dr\,d\theta}$$

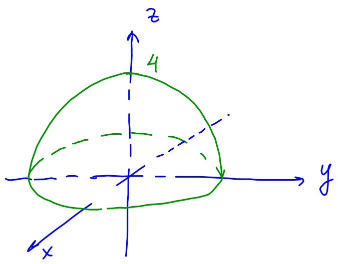
we will get

$$\boxed{\iiint_E f(x, y, z) dV = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} \int_{u_1(r\cos\theta, r\sin\theta)}^{u_2(r\cos\theta, r\sin\theta)} f(r\cos\theta, r\sin\theta, z)\, r\, dz\, dr\, d\theta}$$

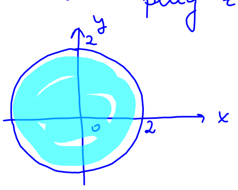
Example 3. Sketch the solid whose volume is given by $\int_0^{2\pi} \int_0^2 \int_0^{4-r^2} r\,dz\,dr\,d\theta$.

$0 \leq \theta \leq 2\pi$
$0 \leq r \leq 2$
$0 \leq z \leq 4 - r^2$

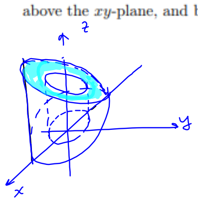
$z = 0$ (xy)-plane
$z = 4 - r^2$, $r^2 = x^2 + y^2$
$z = 4 - (x^2 + y^2)$ - elliptic paraboloid



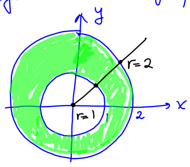
Projection onto the (xy) plane
plug $z = 0$: $0 = 4 - (x^2 + y^2)$
$x^2 + y^2 = 4$.
$x = r\cos\theta$
$y = r\sin\theta$
$0 \leq r \leq 2$, $0 \leq \theta \leq 2\pi$.



Example 4. Evaluate $\iiint_E y\,dV$ where E is the solid that lies between the cylinders $x^2+y^2=1$ and $x^2+y^2=4$, above the xy-plane, and below the plane $z=x+2$.



Projection onto (xy)-plane



Cylindrical coordinates

$x = r\cos\theta$

$y = r\sin\theta$

$z = z$

$dV = r\,dz\,dr\,d\theta$

$0 \le z \le x+2$

$0 \le z \le r\cos\theta + 2$

$1 \le r \le 2\ ,\quad 0 \le \theta \le 2\pi$

$$\iiint_E y\,dV = \int_0^{2\pi}\int_1^2\int_0^{r\cos\theta+2} r\sin\theta\, r\,dz\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_1^2 r^2\sin\theta\, z\Big|_{z=0}^{z=r\cos\theta+2} dr\,d\theta$$

$$= \int_0^{2\pi}\int_1^2 r^2\sin\theta\,(r\cos\theta+2)\,dr\,d\theta = \int_0^{2\pi}\int_1^2 \left(r^3\sin\theta\cos\theta + 2r^2\sin\theta\right)dr\,d\theta$$

$$= \int_0^{2\pi}\left[\frac{r^4}{4}\sin\theta\cos\theta + \frac{2r^3}{3}\sin\theta\right]_1^2 d\theta$$

$$= \int_0^{2\pi}\left[\frac{15}{4}\sin\theta\cos\theta + \frac{14}{3}\sin\theta\right]d\theta = \frac{15}{4}\underbrace{\int_0^{2\pi}\sin\theta\cos\theta\,d\theta}_{0} + \frac{14}{3}\underbrace{\int_0^{2\pi}\sin\theta\,d\theta}_{0} = \boxed{0}$$

$u = \sin\theta$

$du = \cos\theta\,d\theta$

$0 \le u \le 0$