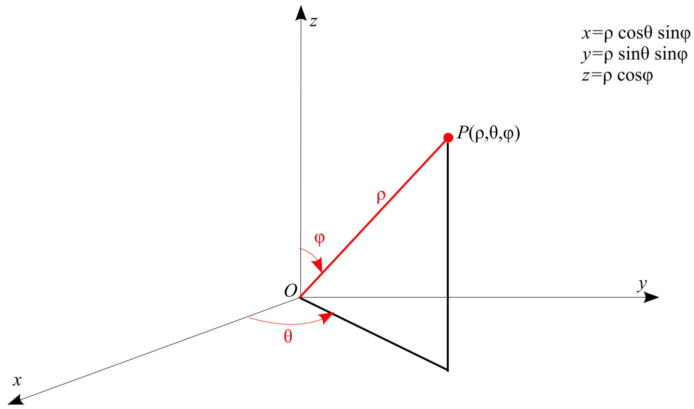
Section 15.8 **Triple integrals spherical coordinates.**

Spherical coordinate system:



To convert from rectangular to spherical coordinates we use

$$\rho^2 = x^2 + y^2 + z^2 \qquad \cos\varphi = \frac{z}{\rho} \qquad \cos\theta = \frac{x}{\rho\sin\varphi}$$

Example 1.

1. The point $(1, \pi/4, \pi/6)$ is given in spherical coordinates. Find its rectangular coordinates.

2. The point $(-\sqrt{3}, -3, -2)$ is given in rectangular coordinates. Find its spherical coordinates.

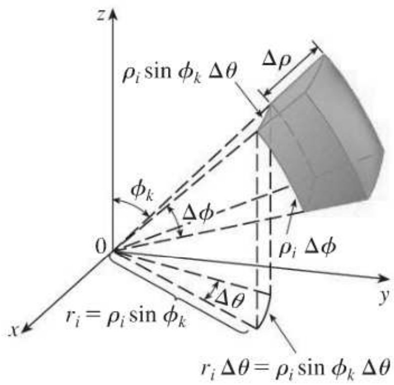
Example 2. Sketch the solid described by the inequalities

$$-\pi/2 \le \theta \le \pi/2, \quad 0 \le \varphi \le \pi/6, \quad 0 \le \rho \le \sec\varphi$$

In this coordinate system the analog of rectangular box is a **spherical wedge**

$$E = \{(\rho, \theta, \varphi) | a \le \rho \le b, \alpha \le \theta \le \beta, \gamma \le \varphi \le \delta\}$$

where $a \ge 0$, $\beta - \alpha \le 2\pi$, and $\delta - \gamma \le \pi$.



The Jacobian of the transformation is

$$\frac{\partial(x,y,z)}{\partial(\rho,\varphi,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \varphi} \\ \frac{\partial y}{\partial \rho} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \varphi} \\ \frac{\partial z}{\partial \rho} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \varphi} \end{vmatrix}$$

Thus,

$$\boxed{\iiint\limits_E f(x,y,z)dV = \int_\gamma^\delta \int_\alpha^\beta \int_a^b f(\rho\cos\theta\sin\varphi, \rho\sin\theta\sin\varphi, \rho\cos\varphi)\rho^2 \sin\varphi \, d\rho \, d\theta \, d\varphi}$$

where $E = \{(\rho, \theta, \varphi) | a \le \rho \le b, \alpha \le \theta \le \beta, \gamma \le \varphi \le \delta\}$

This formula can be extended to include more general spherical regions such as

$$E = \{(\rho, \theta, \varphi) | \alpha \le \theta \le \beta, \gamma \le \varphi \le \delta \, , g_1(\theta, \varphi) \le \rho \le g_2(\theta, \varphi)\}$$

In this case

$$\boxed{\iiint\limits_E f(x,y,z)dV = \int_\gamma^\delta \int_\alpha^\beta \int_{g_1(\theta,\varphi)}^{g_2(\theta,\varphi)} f(\rho\cos\theta\sin\varphi, \rho\sin\theta\sin\varphi, \rho\cos\varphi)\rho^2 \sin\varphi \, d\rho \, d\theta \, d\varphi}$$

Example 3. Find the volume of the solid E that lies above the cone $z = \sqrt{x^2 + y^2}$ and below the sphere $x^2 + y^2 + z^2 = 1$.