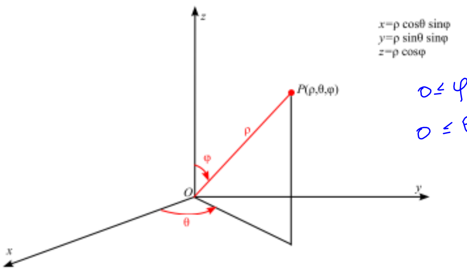
Section 15.8 **Triple integrals in spherical coordinates.**

Spherical coordinate system:



$0 \le \varphi \le \pi$

$0 \le \theta \le 2\pi$

To convert from rectangular to spherical coordinates we use

$$\rho^2 = x^2 + y^2 + z^2 \qquad \cos\varphi = \frac{z}{\rho} \qquad \cos\theta = \frac{x}{\rho\sin\varphi}$$

Example 1.

1. The point $(\underbrace{1}_{\rho}, \underbrace{\pi/4}_{\theta}, \underbrace{\pi/6}_{\varphi})$ is given in spherical coordinates. Find its rectangular coordinates.

$x = \rho\cos\theta\sin\varphi = (1)\cos\frac{\pi}{4}\sin\frac{\pi}{6} = \frac{\sqrt{2}}{2}\cdot\frac{1}{2} = \frac{\sqrt{2}}{4}$

$y = \rho\sin\theta\sin\varphi = \frac{\sqrt{2}}{2}\,\frac{1}{2} = \frac{\sqrt{2}}{4}$

$z = \rho\cos\varphi = (1)\cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$

$\boxed{\left(\frac{\sqrt{2}}{4}, \frac{\sqrt{2}}{4}, \frac{\sqrt{3}}{2}\right)}$

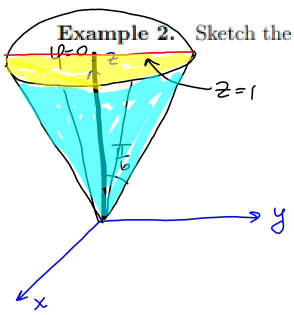
2. The point $(-\sqrt{3}, -3, -2)$ is given in rectangular coordinates. Find its spherical coordinates.

$\rho^2 = x^2 + y^2 + z^2 = 3 + 9 + 4 = 16, \quad \rho = 4$

$\cos\varphi = \frac{z}{\rho} = -\frac{2}{4} = -\frac{1}{2}, \quad \varphi = \frac{2\pi}{3}$

$\cos\theta = \frac{x}{\rho\sin\varphi} = \frac{-\sqrt{3}}{4\sin\frac{2\pi}{3}} = \frac{-\sqrt{3}}{4\frac{\sqrt{3}}{2}} = -\frac{1}{2}, \quad \theta = \frac{4\pi}{3}$

$\boxed{\left(4, \frac{4\pi}{3}, \frac{2\pi}{3}\right)}$



Example 2. Sketch the solid described by the inequalities

$$\underbrace{-\pi/2 \le \theta \le \pi/2}_{x \ge 0}, \quad 0 \le \varphi \le \pi/6, \quad 0 \le \rho \le \sec\varphi$$

$\varphi = 0$ - positive z-axis

$\varphi = \frac{\pi}{6}$ - cone

$\rho = 0$ is the origin

$\rho = \sec\varphi$

$(\cos\varphi)\rho = \frac{1}{\cos\varphi}(\cos\varphi)$

$\underbrace{\rho\cos\varphi}_{z} = 1$ plane $z = 1$

In this coordinate system the analog of rectangular box is a **spherical wedge**

$$E = \{(\rho, \theta, \varphi) | a \le \rho \le b, \alpha \le \theta \le \beta, \gamma \le \varphi \le \delta\}$$

where $a \ge 0$, $\beta - \alpha \le 2\pi$, and $\delta - \gamma \le \pi$.

$\iiint\limits_E f(x,y,z)\,dV$

spherical coordinates

$$\begin{cases} x = \rho\cos\theta\sin\varphi \\ y = \rho\sin\theta\sin\varphi \\ z = \rho\cos\varphi \end{cases}$$

$$dV = \left|\frac{\partial(x,y,z)}{\partial(\rho,\theta,\varphi)}\right| d\rho\, d\varphi\, d\theta$$

$x = \rho\cos\theta\sin\varphi$

$\frac{\partial x}{\partial \rho} = \cos\theta\sin\varphi$

$\frac{\partial x}{\partial \theta} = -\rho\sin\theta\sin\varphi$

$\frac{\partial x}{\partial \varphi} = \rho\cos\theta\cos\varphi$

$y = \rho\sin\theta\sin\varphi$

$\frac{\partial y}{\partial \rho} = \sin\theta\sin\varphi$

$\frac{\partial y}{\partial \theta} = \rho\cos\theta\sin\varphi$

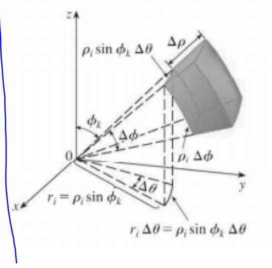
$\frac{\partial y}{\partial \varphi} = \rho\sin\theta\cos\varphi$

$z = \rho\cos\varphi$

$\frac{\partial z}{\partial \rho} = \cos\varphi$

$\frac{\partial z}{\partial \theta} = 0$

$\frac{\partial z}{\partial \varphi} = -\rho\sin\varphi$



The Jacobian of the transformation is

$$\frac{\partial(x,y,z)}{\partial(\rho,\theta,\varphi)} = \begin{vmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \varphi} \\ \frac{\partial y}{\partial \rho} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \varphi} \\ \frac{\partial z}{\partial \rho} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \varphi} \end{vmatrix} = \begin{vmatrix} \cos\theta\sin\varphi & -\rho\sin\theta\sin\varphi & \rho\cos\theta\cos\varphi \\ \sin\theta\sin\varphi & \rho\cos\theta\sin\varphi & \rho\sin\theta\cos\varphi \\ \cos\varphi & 0 & -\rho\sin\varphi \end{vmatrix}$$

$$= -\rho^2\cos^2\theta\sin^3\varphi - \rho^2\sin^2\theta\sin\varphi\cos^2\varphi - \rho^2\cos^2\theta\sin\varphi\cos^2\varphi - \rho^2\sin^2\theta\sin^3\varphi$$

$$= -\rho^2\left[\sin^3\varphi\,(\underbrace{\cancel{\cos^2\theta + \sin^2\theta}}_{1}) + \sin\varphi\cos^2\varphi\,(\underbrace{\cancel{\sin^2\theta + \cos^2\theta}}_{1})\right]$$

$$= -\rho^2\sin\varphi\,[\underbrace{\cancel{\sin^2\varphi + \cos^2\varphi}}_{1}] = -\rho^2\sin\varphi$$

$$dV = \left|\frac{\partial(x,y,z)}{\partial(\rho,\theta,\varphi)}\right| d\rho\, d\varphi\, d\theta = \rho^2\sin\varphi\, d\rho\, d\varphi\, d\theta$$

$$\boxed{dV = \rho^2\sin\varphi\, d\rho\, d\varphi\, d\theta}$$

Thus,

$$\boxed{\iiint\limits_E f(x,y,z)\,dV = \int_\gamma^\delta \int_\alpha^\beta \int_a^b f(\overbrace{\rho\cos\theta\sin\varphi}^{x}, \overbrace{\rho\sin\theta\sin\varphi}^{y}, \overbrace{\rho\cos\varphi}^{z})\overbrace{\rho^2\sin\varphi\, d\rho\, d\theta\, d\varphi}^{dV}}$$

where $E = \{(\rho, \theta, \varphi) | a \le \rho \le b, \alpha \le \theta \le \beta, \gamma \le \varphi \le \delta\}$

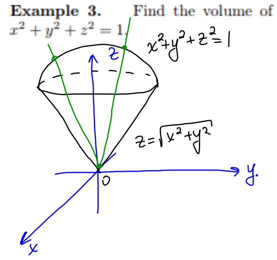
This formula can be extended to include more general spherical regions such as

$$E = \{(\rho, \theta, \varphi) | \alpha \le \theta \le \beta, \gamma \le \varphi \le \delta, g_1(\theta,\varphi) \le \rho \le g_2(\theta,\varphi)\}$$

In this case

$$\boxed{\iiint\limits_E f(x,y,z)\,dV = \int_\gamma^\delta \int_\alpha^\beta \int_{g_1(\theta,\varphi)}^{g_2(\theta,\varphi)} f(\rho\cos\theta\sin\varphi, \rho\sin\theta\sin\varphi, \rho\cos\varphi)\rho^2\sin\varphi\, d\rho\, d\theta\, d\varphi}$$

Example 3. Find the volume of the solid E that lies above the cone $z=\sqrt{x^2+y^2}$ and below the sphere $x^2+y^2+z^2=1$



$$\begin{cases} x=\rho\cos\theta\sin\varphi \\ y=\rho\sin\theta\sin\varphi \\ z=\rho\cos\varphi \end{cases}$$

sphere $\quad x^2+y^2+z^2=1$

$\rho^2=1$

or $\boxed{\rho=1}$ sphere

cone $\quad x^2+y^2=\rho^2\cos^2\theta\sin^2\varphi+\rho^2\sin^2\theta\sin^2\varphi$

$=\rho^2\sin^2\varphi(\cos^2\theta+\sin^2\theta)$

$x^2+y^2=\rho^2\sin^2\varphi$

$z=\sqrt{x^2+y^2}$

$\dfrac{\rho\cos\varphi}{\rho\sin\varphi}=\dfrac{\rho\sin\varphi}{\rho\sin\varphi}$

$\cot\varphi=1 \Rightarrow \boxed{\varphi=\frac{\pi}{4}}$

origin $\le\rho\le$ sphere $0\le\rho\le1$	$0\le\varphi\le$ cone $0\le\varphi\le\frac{\pi}{4}$	$0\le\theta\le2\pi$

$$V=\iiint_E 1\,dV=\left|\begin{array}{l} x=\rho\cos\theta\sin\varphi \\ y=\rho\sin\theta\sin\varphi \\ z=\rho\cos\varphi \\ dV=\rho^2\sin\varphi\,d\rho\,d\varphi\,d\theta \end{array}\right|=\int_0^{2\pi}\int_0^{\pi/4}\int_0^1\rho^2\sin\varphi\,d\rho\,d\varphi\,d\theta$$

$$=\int_0^{2\pi}\int_0^{\pi/4}\frac{\rho^3}{3}\Big|_0^1\sin\varphi\,d\varphi\,d\theta=\frac{1}{3}\int_0^{2\pi}\int_0^{\pi/4}\sin\varphi\,d\varphi\,d\theta$$

$$=-\frac{1}{3}\int_0^{2\pi}\cos\varphi\Big|_0^{\pi/4}d\theta=-\frac{1}{3}\int_0^{2\pi}\left(\frac{\sqrt2}{2}-1\right)d\theta$$

$$=-\frac{1}{3}\left(\frac{\sqrt2}{2}-1\right)\int_0^{2\pi}d\theta$$

$$=\boxed{-\frac{2\pi}{3}\left(\frac{\sqrt2}{2}-1\right)}$$