Section 15.9 **Change of variables in multiple intergrals.**

Double integrals.

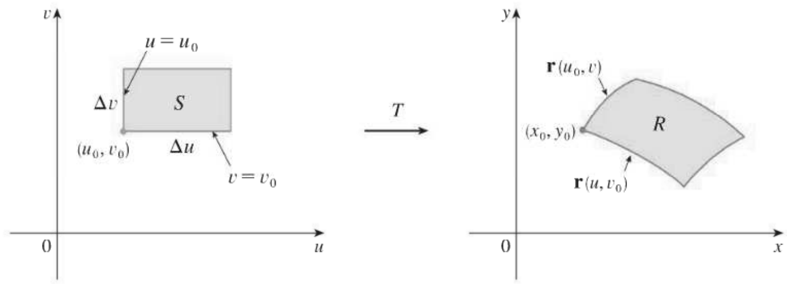
We consider a change of variables that is given by a **transformation** T from the uv-plane to xy-plane:

$$T(u,v) = (x,y) \quad \text{or} \quad x = x(u,v), y = y(u,v)$$

We assume that $\frac{\partial x}{\partial u}$, $\frac{\partial x}{\partial v}$, $\frac{\partial y}{\partial u}$ and $\frac{\partial y}{\partial v}$ are continuous.

Example 1. Let S be the triangular region with vertices (0,0), (1,1), (0,1). Find the image of S under the transformation $x = u^2$, $y = v$.

Now let see how change of variables affects a double integral. We start with a small rectangle S in the uv-plane whose lower left corner is in point (u_0, v_0) and whose dimentions are Δu and Δv.



The image of S is a region R in the xy-plane, one of whose boundary points is $(x_0, y_0) = T(u_0, v_0)$. The vector

$$\mathbf{r}(u, v) = x(u, v)\mathbf{i} + y(u, v)\mathbf{j}$$

is the position vector of the image of the point (u, v). The tangent vector to the image curve $\mathbf{r}(u, v_0)$ at (x_0, y_0) is

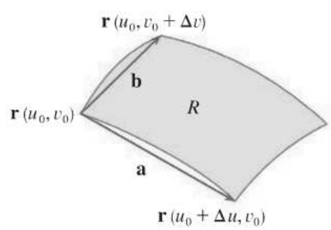
$$\mathbf{r}_u = \frac{\partial x}{\partial u}(u_0, v_0)\mathbf{i} + \frac{\partial y}{\partial u}(u_0, v_0)\mathbf{j}$$

Similarly, the tangent vector to the image curve $\mathbf{r}(u_0, v)$ at (x_0, y_0) is

$$\mathbf{r}_u = \frac{\partial x}{\partial v}(u_0, v_0)\mathbf{i} + \frac{\partial y}{\partial v}(u_0, v_0)\mathbf{j}$$

We can approximate the image region $R = T(S)$ by a parallelogramm determined by the secant vectors

$$\mathbf{a} = \mathbf{r}(u + \Delta u, v_0) - \mathbf{r}(u_0, v_0) \qquad \mathbf{b} = \mathbf{r}(u_0, v + \Delta v) - \mathbf{r}(u_0, v_0)$$



But since

$$\mathbf{r}_u = \lim_{\Delta u \to 0} \frac{\mathbf{r}(u + \Delta u, v_0) - \mathbf{r}(u_0, v_0)}{\Delta u}$$

and

$$\mathbf{r}_u = \lim_{\Delta v \to 0} \frac{\mathbf{r}(u_0, v + \Delta v) - \mathbf{r}(u_0, v_0)}{\Delta v}$$

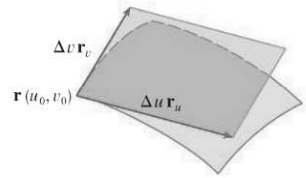
then

$$\mathbf{r}(u + \Delta u, v_0) - \mathbf{r}(u_0, v_0) = \Delta u \ \mathbf{r}_u$$

and

$$\mathbf{r}(u_0, v + \Delta v) - \mathbf{r}(u_0, v_0) = \Delta v \ \mathbf{r}_v$$

Thus, we can approximate R by the parallelogramm determined by the vectors $\Delta u \ \mathbf{r}_u$ and $\Delta v \ \mathbf{r}_v$.



Then the area of R approximately equals to the area of the parallelogramm.

$$A(R) \approx |\mathbf{r}_u \times \mathbf{r}_v| \Delta u \Delta v$$

$$\mathbf{r}_u \times \mathbf{r}_v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} & 0 \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} & 0 \end{vmatrix} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} \mathbf{k}$$

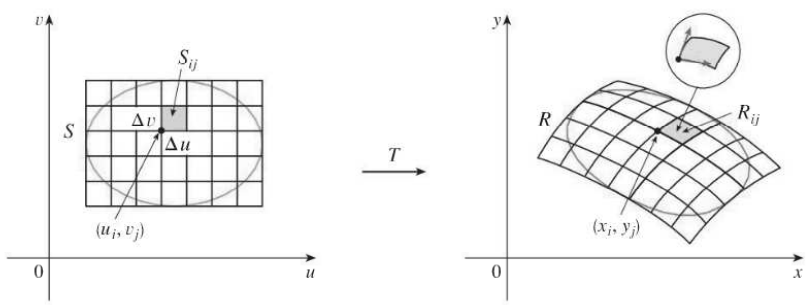
Definition. The **Jacobian** of the transformation T given by $x = x(u, v)$, $y = y(u, v)$ is

$$\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix}$$

We say that

$$\Delta A \approx \left| \frac{\partial(x, y)}{\partial(u, v)} \right| \Delta u \Delta v$$

Next, we divide a region S in the uv-plane into rectanglers S_{ij} and call their images in the xy-plane R_{ij}.



Then

$$\boxed{\iint_R f(x, y) dA = \iint_S f(x(u, v), y(u, v)) \left| \frac{\partial(x, y)}{\partial(u, x)} \right| \, dudv}$$