Section 15.9 **Change of variables in multiple intergrals.**

Double integrals.

We consider a change of variables that is given by a **transformation** T from the uv-plane to xy-plane:

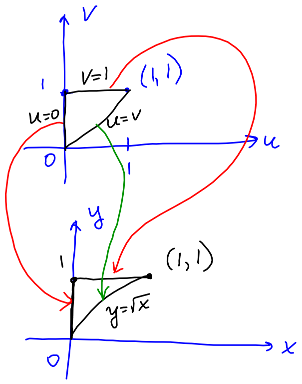
$$T(u,v) = (x,y) \quad \text{or} \quad x = x(u,v), y = y(u,v)$$

We assume that $\dfrac{\partial x}{\partial u}, \dfrac{\partial x}{\partial v}, \dfrac{\partial y}{\partial u}$ and $\dfrac{\partial y}{\partial v}$ are continuous.

Example 1. Let S be the triangular region with vertices (0,0), (1,1), (0,1). Find the image of S under the transformation $x = u^2$, $y = v$.

Original variables (u,v)

New variables (x,y)



Find the image for each side.

$0 \le v \le 1$, $u=0$: $x = u^2 = 0$

$x = 0$, $0 \le y \le 1$

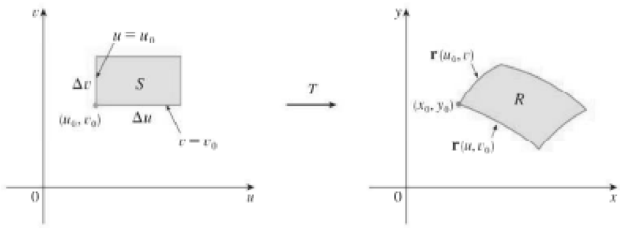
$v = 1$, $0 \le u \le 1$: $y = v = 1$, $x = u^2 \Rightarrow 0 \le x \le 1$

$u = v$, $0 \le v \le 1$, $0 \le u \le 1$.

$\sqrt{x} = y$

$0 \le y \le 1$, $0 \le x \le 1$

Now let see how change of variables affects a double integral. We start with a small rectangle S in the uv-plane whose lower left corner is in point (u_0, v_0) and whose dimentions are Δu and Δv.



The image of S is a region R in the xy-plane, one of whose boundary points is $(x_0, y_0) = T(u_0, v_0)$. The vector

$$\mathbf{r}(u,v) = x(u,v)\mathbf{i} + y(u,v)\mathbf{j}$$

is the position vector of the image of the point (u,v). The tangent vector to the image curve $\mathbf{r}(u, v_0)$ at (x_0, y_0) is

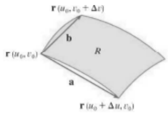
$$\mathbf{r}_u = \frac{\partial x}{\partial u}(u_0, v_0)\mathbf{i} + \frac{\partial y}{\partial u}(u_0, v_0)\mathbf{j}$$

Similarly, the tangent vector to the image curve $\mathbf{r}(u_0, v)$ at (x_0, y_0) is

$$\mathbf{r}_v = \frac{\partial x}{\partial v}(u_0, v_0)\mathbf{i} + \frac{\partial y}{\partial v}(u_0, v_0)\mathbf{j}$$

We can approximate the image region $R = T(S)$ by a parallelogramm determined by the secant vectors

$$\mathbf{a} = \mathbf{r}(u + \Delta u, v_0) - \mathbf{r}(u_0, v_0) \qquad \mathbf{b} = \mathbf{r}(u_0, v + \Delta v) - \mathbf{r}(u_0, v_0)$$



But since

$$\mathbf{r}_u = \lim_{\Delta u \to 0} \frac{\mathbf{r}(u + \Delta u, v_0) - \mathbf{r}(u_0, v_0)}{\Delta u}$$

and

$$\mathbf{r}_u = \lim_{\Delta v \to 0} \frac{\mathbf{r}(u_0, v + \Delta v) - \mathbf{r}(u_0, v_0)}{\Delta v}$$

then

$$\mathbf{r}(u + \Delta u, v_0) - \mathbf{r}(u_0, v_0) = \Delta u \, \mathbf{r}_u$$

and

$$\mathbf{r}(u_0, v + \Delta v) - \mathbf{r}(u_0, v_0) = \Delta v \, \mathbf{r}_v$$

Thus, we can approximate R by the parallelogramm determined by the vectors $\Delta u \, \mathbf{r}_u$ and $\Delta v \, \mathbf{r}_v$.

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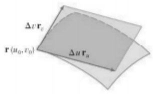


Then the area of R approximately equals to the area of the parallelogramm.

$$\boxed{A(R) \approx |\mathbf{r}_u \times \mathbf{r}_v| \Delta u \Delta v}$$

$$\mathbf{r}_u \times \mathbf{r}_v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & 0 \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & 0 \end{vmatrix} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} \mathbf{k}$$

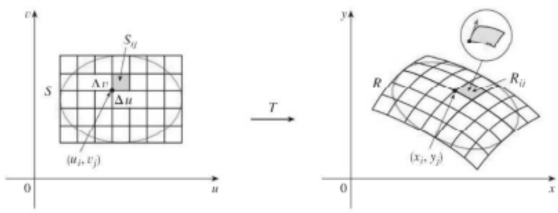
Definition. The Jacobian of the transformation T given by $x = x(u, v)$, $y = y(u, v)$ is

$$\boxed{\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}}$$

We say that

$$\Delta A \approx \left| \frac{\partial(x, y)}{\partial(u, v)} \right| \Delta u \Delta v$$

Next, we divide a region S in the uv-plane into rectanglers S_{ij} and call their images in the xy-plane R_{ij}.



Then

$$\boxed{\iint_R f(x, y) dA = \iint_S f(x(u, v), y(u, v)) \left| \frac{\partial(x, y)}{\partial(u, x)} \right| dudv}$$

$x = x(u,v)$
$y = y(u,v)$
$dA = \left| \frac{\partial(x,y)}{\partial(u,v)} \right| dudv$

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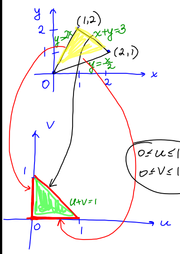
Example 2 Use the transformation
$x = 2u + v$
$y = u + 2v$
to evaluate the integral $\iint_R (x-3y)\,dA$, where R is the triangular region with vertices $(0,0)$, $(2,1)$ and $(1,2)$.



$0 \le u \le 1$
$0 \le v \le 1-u$

Jacobian:

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} = 4-1=3$$

Express u, v in terms of x and y

$$\begin{cases} x = 2u+v \\ y = u+2v \end{cases} \Rightarrow v = x-2u$$

$$y = u + 2(x-2u)$$

$$y = u + 2x - 4u \Rightarrow y = 2x - 3u$$

$$\boxed{u = \frac{2x-y}{3}}$$

$$v = x - 2u = x - \frac{4x-2y}{3}$$

$$\boxed{v = \frac{-x+2y}{3}}$$

(x,y)		(u,v)	
$(0,0)$	$\longrightarrow$	$u = \frac{2(0)-0}{3} = 0,\ v = \frac{-0+2(0)}{3} = 0$	$(0,0)$
$(1,2)$	$\longrightarrow$	$u = \frac{2(1)-2}{3} = 0,\ v = \frac{-(1)+2(2)}{3} = 1$	$(0,1)$
$(2,1)$	$\longrightarrow$	$u = \frac{4-1}{3} = 1,\ v = \frac{-2+2}{3} = 0$	$(1,0)$

$y = 2x$: $\quad \underbrace{u+2v}_{y} = 2\underbrace{(2u+v)}_{x} \Rightarrow u + \cancel{2v} = 4u + \cancel{2v} \Rightarrow 3u = 0$ or $\boxed{u=0}$

$y = \frac{x}{2}$: $\quad u+2v = \frac{2u+v}{2} \Rightarrow \cancel{u} + 2v = \cancel{u} + \frac{v}{2} \Rightarrow \frac{3v}{2} = 0$ or $\boxed{v=0}$

$x+y=3$: $\quad (2u+v)+(u+2v) = 3 \Rightarrow 3u+3v=3$ or $\boxed{u+v=1}$

$$\iint_R (x-3y)\,dA = \int_0^1\int_0^{1-u} \left[2u+v-3(u+2v)\right](3)\,dv\,du$$

$$= -3\int_0^1\int_0^{1-u} \left[+u+5v\right]dv\,du = -3\int_0^1 \left[uv + \frac{5v^2}{2}\right]_{v=0}^{v=1-u} du$$

$$= -3\int_0^1 \left[u(1-u) + \frac{5}{2}(1-u)^2\right] du$$

$$= -3\int_0^1 \left[u-u^2+\frac{5}{2}(1-2u+u^2)\right] du = -3\int_0^1 \left[u-u^2+\frac{5}{2}-5u+\frac{5}{2}u^2\right] du$$

$$= -3\int_0^1 \left[\frac{5}{2} - 4u + \frac{3}{2}u^2\right] du$$

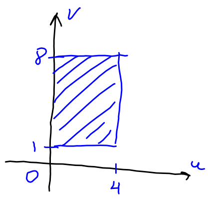
$$= -3\left[\frac{5}{2}u - 2u^2 + \frac{u^3}{2}\right]_0^1 = -3\left[\frac{5}{2} - 2 + \frac{1}{2}\right] = \boxed{-3}$$

E.3. Evaluate the integral by making an appropriate change of variables.

$\iint_R \frac{x-2y}{3x-y}\,dA$, where R is the parallelogram enclosed by the lines $x-2y=0$, $x-2y=4$, $3x-y=1$, $3x-y=8$.

$u = x-2y$
$v = 3x-y$

Region of integration: $0 \le u \le 4$, $1 \le v \le 8$



Jacobian $\quad u = x-2y \Rightarrow x = u+2y$

$$v = 3(u+2y) - y$$

$$v = 3u + 6y - y$$

$$v = 3u + 5y \Rightarrow y = \frac{v-3u}{5} = \boxed{\frac{1}{5}v - \frac{3}{5}u = y}$$

$$x = u + 2\left[\frac{1}{5}v - \frac{3}{5}u\right] = u + \frac{2}{5}v - \frac{6}{5}u$$

$$\boxed{x = \frac{2}{5}v - \frac{1}{5}u}$$

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} -\frac{1}{5} & \frac{2}{5} \\ -\frac{3}{5} & \frac{1}{5} \end{vmatrix} = -\frac{1}{25} + \frac{6}{25} = \frac{5}{25} = \frac{1}{5}$$

$$\iint_R \frac{x-2y}{3x-y}\,dA = \int_1^8\int_0^4 \frac{u}{v}\left(\frac{1}{5}\right)du\,dv = \frac{1}{5}\int_1^8 \frac{1}{v}\cdot\frac{u^2}{2}\Big|_0^4\,dv$$

$$= \frac{8}{5}\int_1^8 \frac{1}{v}\,dv = \frac{8}{5}\ln|v|\Big|_1^8 = \frac{8}{5}\left(\ln 8 - \cancel{\ln 1}^{\,0}\right) = \boxed{\frac{8\ln 8}{5}}$$

Triple integrals.

$x = x(u,v,w)$
$y = y(u,v,w)$
$z = z(u,v,w)$

$$\boxed{\iiint_E f(x,y,z)\,dV = \iiint_S f\big(x(u,v,w), y(u,v,w), z(u,v,w)\big)\left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| du\,dv\,dw}$$

where

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$