Chapter 16. **Vector calculus.**
Section 16.1 **Vector fields.**

Definition. Let D be a set in $\mathbb{R}^2$ (a plane region). A **vector field on** $\mathbb{R}^2$ is a function $\mathbf{F}$ that assigns to each point $(x, y) \in D$ a two-dimensional vector $\mathbf{F}(x, y)$.

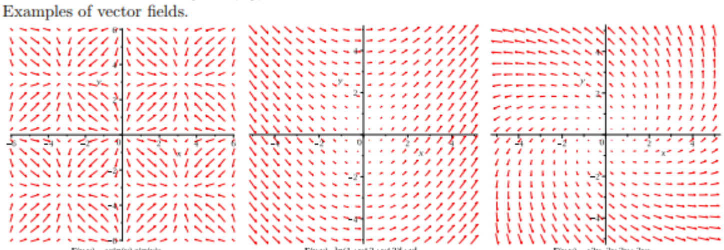
$$\mathbf{F}(x, y) = P(x, y)\mathbf{i} + Q(x, y)\mathbf{j}$$

The **components functions** P and Q are sometimes called **scalar fields**.

Definition. Let E be a set in $\mathbb{R}^3$. A **vector field on** $\mathbb{R}^3$ is a function $\mathbf{F}$ that assigns to each point $(x, y, z) \in E$ a three-dimensional vector $\mathbf{F}(x, y, z)$.

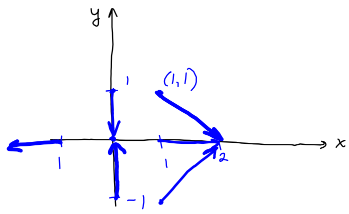
$$\mathbf{F}(x, y, z) = P(x, y, z)\mathbf{i} + Q(x, y, z)\mathbf{j} + R(x, y, z)\mathbf{k}$$

$\mathbf{F}$ is continuous is and only if P, Q, and R are continuous.
Examples of vector fields.



Example 1. Sketch the vector field $\mathbf{F}$ if $\mathbf{F}(x, y) = x\mathbf{i} - y\mathbf{j}$. $= \langle x, -y \rangle$

$(0,0)$	$\vec{F}(0,0) = \langle 0,0 \rangle$
$(0,1)$	$\vec{F}(0,1) = \langle 0,-1 \rangle$
$(1,0)$	$\vec{F}(1,0) = \langle 1, 0 \rangle$
$(0,-1)$	$\vec{F}(0,-1) = \langle 0, 1 \rangle$
$(-1,0)$	$\vec{F}(-1,0) = \langle -1, 0 \rangle$
$(1,1)$	$\vec{F}(1,1) = \langle 1,-1 \rangle$
$(1,-1)$	$\vec{F}(1,-1) = \langle 1, 1 \rangle$



Let $f(x, y)$ be a scalar function of two variables, then

$$\nabla f(x, y) = \langle f_x, f_y \rangle$$

is a vector field called a **gradient vector field**.
If $f(x, y, z)$ be a scalar function of three variables, then its gradient vector field is defined as

$$\nabla f(x, y, z) = \langle f_x, f_y, f_z \rangle$$

Example 2. Find the gradient vector field of the function $f(x, y, z) = x \ln(y - z)$.

$$\nabla f = \langle f_x, f_y, f_z \rangle = \boxed{\left\langle \ln(y-z), \frac{x}{y-z}, -\frac{x}{y-z} \right\rangle}$$

A vector field is called a **conservative vector field** if it is the gradient of some scalar function, that is, if there exists a function f such that $\mathbf{F} = \nabla f$. Then f is called a **potential function** for $\mathbf{F}$.