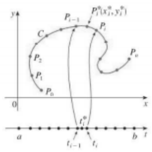
Section 16.2 **Line integrals.**

Let C be a smooth plane curve with parametric equations

$$x = x(t), \qquad y = y(t), \qquad a \le t \le b$$



A partition of the parameter interval $[a, b]$ by points t_i with

$$a = t_0 < t_1 < ... < t_n = b$$

determine a partition P of the curve by points $P_i(x_i, y_i)$, where $x_i = x(t_i)$, $y_i = y(t_i)$, $z_i = z(t_i)$. Points P_i divide C into n subarcs with length Δs_1, Δs_2,...,Δs_n. The **norm** $\|P\|$ of the partition is the longest of these lengths. We choose any point $P_i^*(x_i^*, y_i^*)$ in the ith subarc.

Definition. If f is defined on a smooth curve C given by

$$x = x(t), \qquad y = y(t), \qquad a \le t \le b$$

then the line integral of f along C with respect to arc length is

$$\int_C f(x, y)ds = \lim_{\|P\| \to 0} \sum_{i=1}^{n} f(x_i^*, y_i^*)\Delta s_i$$

if this limit exists.

Since

$$ds = \sqrt{\left[\frac{dx}{dt}\right]^2 + \left[\frac{dy}{dt}\right]^2}\, dt$$

then

$$\boxed{\int_C f(x, y)ds = \int_a^b f(x(t), y(t))\sqrt{\left[\frac{dx}{dt}\right]^2 + \left[\frac{dy}{dt}\right]^2}\, dt}$$

The value of the line integral does not depend on the parametrization of the curve provided that the curve is traversed exactly once as t increases from a to b.

Example 1. Evaluate the line integral $\int_C x ds$, where C is a given by $x = t^3$, $y = t$, $0 \le t \le 1$.

$x = t^3, \quad x'(t) = 3t^2$

$y = t, \quad y'(t) = 1$

$ds = \sqrt{[x'(t)]^2 + [y'(t)]^2}\, dt = \sqrt{9t^4+1}\, dt$

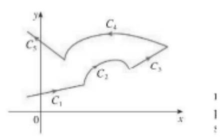
$\int_C x\, ds = \int_0^1 \overset{x}{t^3} \underbrace{\sqrt{9t^4+1}\, dt}_{ds}$

$u = 9t^4+1$, $du = 36t^3\, dt$, $1 \le u \le 10$

$= \frac{1}{36}\int_1^{10} \sqrt{u}\, du = \frac{1}{36} \frac{u^{3/2}}{3/2}\Big|_1^{10}$

$= \frac{1}{54}\left(10^{3/2} - 1\right) = \boxed{\frac{10\sqrt{10} - 1}{54}}$

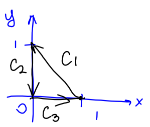
Let C be a piecewise-smooth curve; that is, C is a union of a finite number of smooth curves C_1, C_2,...,C_n.



Then

$$\boxed{\int_C f(x, y)ds = \int_{C_1} f(x, y)ds + \int_{C_2} f(x, y)ds + ... + \int_{C_n} f(x, y)ds}$$

Example 2. Evaluate $\int_C (x + y)ds$ if C consists of line segments from (1,0) to (0,1), from (0,1) to (0,0), and from (0,0) to ~~(0,1)~~ (1,0)



$\int_C = \int_{C_1} + \int_{C_2} + \int_{C_3}$

C_1: $x + y = 1 \Rightarrow x = 1 - y$, $0 \le y \le 1$

$x = x$

$ds = \sqrt{1 + [x'(y)]^2}\, dy = \sqrt{1 + (-1)^2}\, dy = \sqrt{2}\, dy$

$\int_{C_1} (x+y) ds = \int_0^1 1 \sqrt{2}\, dy = \sqrt{2}\, y\Big|_0^1 = \sqrt{2}$

C_2: $x = 0$, $y = y$, $1 \le y \le 0$

$ds = \sqrt{1 + [x'(y)]^2}\, dy = dy$

$\int_{C_2} (x+y) ds = \int_1^0 (0 + y)\, dy = \frac{y^2}{2}\Big|_1^0 = -\frac{1}{2}$

C_3: $y = 0$, $x = x$, $0 \le x \le 1$, $ds = \sqrt{1 + [y'(x)]^2}\, dx = dx$

$\int_{C_3} (x+y) ds = \int_0^1 (x + 0)\, dx = \frac{x^2}{2}\Big|_0^1 = \frac{1}{2}$

$\int_C = \int_{C_1} + \int_{C_2} + \int_{C_3} = \sqrt{2} - \frac{1}{2} + \frac{1}{2} = \boxed{\sqrt{2}}$

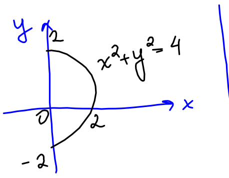
Physical interpretation of a line integral $\int_C f(x,y)ds$**.** Suppose that $\rho(x,y)$ represents the linear density at a point (x,y) of a thin wire shaped like a curve C. Then the **mass** of wire is

$$\boxed{m = \int_C \rho(x,y)ds}$$

The **center of mass** of the wire with density function ρ is at the point $(\bar{x},\bar{y})$, where

$$\boxed{\bar{x} = \frac{1}{m}\int_C x\rho(x,y)ds} \qquad \boxed{\bar{y} = \frac{1}{m}\int_C y\rho(x,y)ds}$$

Example 3. A thin wire is bent into the shape of a semicircle $x^2+y^2=4$, $x \geq 0$. If the linear density is a constant k, find the mass and center of mass of the wire.



$x = 2\cos t$, $y = 2\sin t$, $-\frac{\pi}{2} \leq t \leq \frac{\pi}{2}$

$\rho(x,y) = k$ - const.

$ds = \sqrt{[x'(t)]^2 + [y'(t)]^2}\,dt = \sqrt{4\sin^2 t + 4\cos^2 t}\,dt = 2dt$

mass $m = \int_C \rho(x,y)ds = \int_{-\pi/2}^{\pi/2} k\,2dt = 2k\,t\Big|_{-\pi/2}^{\pi/2} = \boxed{2\pi k}$

Center of mass: $\bar{y} = 0$ (lies on the x-axis).

$\bar{x} = \frac{1}{m}\int_C x\rho(x,y)ds = \frac{1}{2\pi \cancel{k}}\int_{-\pi/2}^{\pi/2} \cancel{2}\cos t(\cancel{k})\,2dt = \frac{2}{\pi}\int_{-\pi/2}^{\pi/2}\cos t\,dt$

$= \frac{2}{\pi}(\sin t)\Big|_{-\pi/2}^{\pi/2} = \frac{4}{\pi}$

$\boxed{\left(\frac{4}{\pi}, 0\right)}$

Line integrals of f along C with respect to x and y:

$$\int_C f(x,y)dx = \lim_{\|P\|\to 0}\sum_{i=1}^{n} f(x_i^*,y_i^*)\Delta x_i, \qquad \Delta x_i = x_i - x_{i-1}$$

$$\int_C f(x,y)dy = \lim_{\|P\|\to 0}\sum_{i=1}^{n} f(x_i^*,y_i^*)\Delta y_i, \qquad \Delta y_i = y_i - y_{i-1}$$

If $x = x(t)$, $y = y(t)$, then $dx = x'(t)dt$, $dy = y'(t)dt$, and

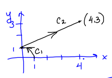
$$\boxed{\int_C f(x,y)dx = \int_a^b f(x(t),y(t))x'(t)dt}$$

$$\boxed{\int_C f(x,y)dy = \int_a^b f(x(t),y(t))y'(t)dt}$$

In general, we will write

$$\int_C P(x,y)dx + Q(x,y)dy = \int_C P(x,y)dx + \int_C Q(x,y)dy$$

Example 3. Evaluate $\int_C x\sqrt{y}dx + 2y\sqrt{x}dy$, if C consists of the arc of the circle $x^2+y^2=1$ from (1,0) to (0,1) and the line segment from (0,1) to (4,3).



$\int_C = \int_{C_1} + \int_{C_2}$

C_1: $x = \cos t$, $y = \sin t$, $0 \leq t \leq \frac{\pi}{2}$, $dx = x'(t)dt = -\sin t\,dt$, $dy = y'(t)dt = \cos t\,dt$

$\int_{C_1} x\sqrt{y}\,dx + 2y\sqrt{x}\,dy = \int_0^{\pi/2}\Big[\underbrace{\cos t}_{x}\underbrace{\sqrt{\sin t}}_{\sqrt{y}}\underbrace{(-\sin t)dt}_{dx} + 2\underbrace{\sin t}_{y}\underbrace{\sqrt{\cos t}}_{\sqrt{x}}\underbrace{\cos t\,dt}_{dy}\Big]$

$= -\int_0^{\pi/2}\cos t(\sin t)^{3/2}dt + 2\int_0^{\pi/2}\sin t(\cos t)^{3/2}dt$

$u = \sin t$, $du = \cos t\,dt$, $0 \leq u \leq 1$ | $v = \cos t$, $dv = -\sin t\,dt$, $1 \leq v \leq 0$

$= -\int_0^1 u^{3/2}du - 2\int_1^0 v^{3/2}dv = -\frac{u^{5/2}}{5/2}\Big|_0^1 - 2\frac{v^{5/2}}{5/2}\Big|_1^0 = -\frac{2}{5} + \frac{4}{5} = \frac{2}{5}$

C_2: line through (0,1) and (4,3).

$\frac{x-0}{4-0} = \frac{y-1}{3-1} \Rightarrow \frac{x}{4} = \frac{y-1}{2}$ or $x = 2y-2$, $y = y$, $1 \leq y \leq 3$. | $dx = x'(y)dy = 2dy$

$\int_{C_2} x\sqrt{y}\,dx + 2y\sqrt{x}\,dy = \int_1^3\left[(2y-2)\sqrt{y}(2dy) + 2y\sqrt{2y-2}\,dy\right]$

$= 4\int_1^3(y^{3/2} - \sqrt{y})dy + \int_1^3 2y\sqrt{2y-2}\,dy$ $\quad$ $2y-2=u$, $2y = u+2$, $du = 2dy$, $0 \leq u \leq 4$

$= 4\int_1^3(y^{3/2} - \sqrt{y})dy + \int_0^4 (u+2)\sqrt{u}\frac{du}{2} = 4\left[\frac{y^{5/2}}{5/2} - \frac{y^{3/2}}{3/2}\right]_1^3 + \frac{1}{2}\int_0^4(u^{3/2} + 2\sqrt{u})du$

$= 8\left[\frac{3^{5/2}-1}{5} - \frac{3^{3/2}-1}{3}\right] + \frac{1}{2}\left[\frac{u^{5/2}}{5/2} + 2\frac{u^{3/2}}{3/2}\right]_0^4$

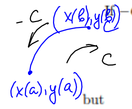
$= 8\left[\frac{3^{5/2}-1}{5} - \frac{3^{3/2}-1}{3}\right] + \frac{1}{5}4^{5/2} + \frac{2}{3}4^{3/2}$

$= 8\left[\frac{3^{5/2}-1}{5} - \frac{3^{3/2}-1}{3}\right] + \frac{32}{5} + \frac{16}{3}$

$\int_C = \int_{C_1} + \int_{C_2} = \boxed{8\left[\frac{3^{5/2}-1}{5} - \frac{3^{3/2}-1}{3}\right] + \frac{32}{5} + \frac{16}{3} + \frac{2}{5}}$

$a \le t \le b$

A given parametrization $x = x(t)$, $y = y(t)$, $a \le t \le b$, determines an **orientation** of a curve C, with the positive direction corresponding to increasing value of the parameter t.



If $-C$ denotes the curve consisting of the same points as C but with the opposite orientation, then we have

$$\boxed{\int_{-C} f(x,y)dx = -\int_C f(x,y)dx} \qquad \boxed{\int_{-C} f(x,y)dy = -\int_C f(x,y)dy}$$

but

$$\boxed{\int_{-C} f(x,y)ds = \int_C f(x,y)ds}$$

Line integrals in space.

Suppose that C is a smooth space curve given by the parametric equations

$$x = x(t), \quad y = y(t) \quad z = z(t), \quad a \le t \le b$$

or by a vector equation $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$. We define the **linear integral of** f **along** C **with respect to arc length** as

$$\boxed{\int_C f(x,y,z)ds = \int_a^b f(x(t),y(t),z(t))\sqrt{\left[\frac{dx}{dt}\right]^2 + \left[\frac{dy}{dt}\right]^2 + \left[\frac{dz}{dt}\right]^2}\,dt = \int_a^b f(\mathbf{r}(t))|\mathbf{r}'(t)|dt}$$

If $f(x,y,z) = 1$, then

$$\int_c ds = \int_a^b |\mathbf{r}'(t)dt| = L$$

Line integral along C with respect to x, y, and z can also be defined as

$$\boxed{\int_C P(x,y,z)dx + Q(x,y,z)dy + R(x,y,z)dz = \int_a^b [P(x,y,z)x'(t) + Q(x,y,z)y'(t) + R(x,y,z)z'(t)]dt}$$

Example 4. Evaluate $\int_C x^2 z ds$ if C is given by $x = \sin(2t)$, $y = 3t$, $z = \cos(2t)$, $0 \le t \le \pi/4$.

$x'(t) = 2\cos(2t)$

$y'(t) = 3$

$z'(t) = -2\sin(2t)$

$$ds = \sqrt{[x'(t)]^2 + [y'(t)]^2 + [z'(t)]^2}\,dt = \sqrt{4\cos^2(2t) + 9 + 4\sin^2(2t)}\,dt$$

$$= \sqrt{4\underbrace{[\cos^2(2t) + \sin^2(2t)]}_{1} + 9}\,dt = \sqrt{13}\,dt$$

$$\int_C x^2 z\, ds = \int_0^{\pi/4} \underbrace{\sin^2(2t)}_{x^2}\underbrace{\cos(2t)}_{z}\underbrace{\sqrt{13}\,dt}_{ds} = \left| \begin{matrix} u = \sin 2t \\ du = 2\cos 2t \\ 0 \le u \le 1 \end{matrix} \right| = \frac{\sqrt{13}}{2}\int_0^1 u^2 du = \frac{\sqrt{13}}{2}\frac{u^3}{3}\Big|_0^1$$

$$= \boxed{\frac{\sqrt{13}}{6}}$$

Example 5. Evaluate $\int_C yzdy + xydz$ if C is given by $x = \sqrt{t}$, $y = t$, $z = t^2$, $0 \le t \le 1$.

$dx = x'(t)dt = \frac{1}{2}t^{-1/2}dt$

$dy = y'(t)dt = dt$

$dz = z'(t)dt = 2t\,dt$

$$\int_C yzdy + xydz = \int_0^1 \left[\underbrace{t}_{y}\underbrace{(t^2)}_{z}\underbrace{dt}_{dy} + \underbrace{\sqrt{t}}_{x}\underbrace{t}_{y}\underbrace{(2t)dt}_{dz}\right]$$

$$= \int_0^1 (t^3 + 2t^{5/2})\,dt = \left[\frac{t^4}{4} + 2\frac{t^{7/2}}{7/2}\right]_0^1 = \boxed{\frac{1}{4} + \frac{4}{7}}$$

Line integrals of vector fields.

Definition. Let $\mathbf{F}$ be continuous vector field defined on a smooth curve C given by a vector function $\mathbf{r}(t)$, $a \le t \le b$. Then the **line integral of** F **along** C is

$$\boxed{\int_C \mathbf{F}\cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t))\cdot \mathbf{r}'(t)dt} = \int_C \mathbf{F}\cdot \mathbf{T}ds$$

where $\mathbf{T}$ is a unit tangent vector.

If $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$, then

$$\boxed{\int_C \mathbf{F}\cdot d\mathbf{r} = \int_C Pdx + Qdy + Rdz}$$

Example 6. Find the work done by the force field $\mathbf{F}(x,y,z) = xz\mathbf{i} + xy\mathbf{j} + yz\mathbf{k}$ on a particle that moves along the curve $\mathbf{r}(t) =< t^2, -t^3, t^4 >$, $0 \le t \le 1$.

$\vec{F}(x,y,z) = \langle xz, xy, yz\rangle$

$\vec{r}(t) = \langle t^2, -t^3, t^4\rangle$, $x(t) = t^2$, $y(t) = -t^3$, $z(t) = t^4$

$\vec{r}\,'(t) = \langle 2t, -3t^2, 4t^3\rangle$

$\vec{F}(\vec{r}(t)) = \langle \overbrace{t^2}^{x}\overbrace{(t^4)}^{z}, \overbrace{t^2}^{x}\overbrace{(-t^3)}^{y}, \overbrace{(-t^3)}^{y}\overbrace{(t^4)}^{z}\rangle$

$= \langle t^6, -t^5, -t^7\rangle$

$\int_C \vec{F}\cdot d\vec{r} = \int_0^1 \vec{F}(\vec{r}(t))\cdot\vec{r}\,'(t)\,dt = \int_0^1 \langle t^6, -t^5, -t^7\rangle\cdot\langle 2t, -3t^2, 4t^3\rangle dt$

$= \int_0^1 [2t^7 + 3t^7 - 4t^{10}]dt = \int_0^1 [5t^7 - 4t^{10}]dt$

$= \left[\frac{5t^8}{8} - \frac{4t^{11}}{11}\right]_0^1 = \boxed{\frac{5}{8} - \frac{4}{11}}$