Section 16.3 **The Fundamental Theorem for line integrals.**

Theorem. Let C be a smooth curve given by the vector function $\mathbf{r}(t)$, $a \le t \le b$. Let f be a differentiable function of two or three variables whose gradient vector ∇f is continuous on C. Then

$$\boxed{\int_C \nabla f \cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a))}$$

Independence of path.

Suppose C_1 and C_2 are two piecewise-smooth curves (which are called **paths**) that have the same initial point A and the terminal point B. In general, $\int_{C_1} \mathbf{F} \cdot d\mathbf{r} \neq \int_{C_2} \mathbf{F} \cdot d\mathbf{r}$. But, according the the Theorem, if ∇f is continuous, then $\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_{C_2} \mathbf{F} \cdot d\mathbf{r}$. In other words, the line integral of a conservative vector field depends only on the initial point and terminal point of a curve.

In general, if $\mathbf{F}$ is a continuous vector-field with domain D, we say that the line integral is **independent of path** if $\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_{C_2} \mathbf{F} \cdot d\mathbf{r}$ for any two paths C_1 and C_2 in D that have the same initial and terminal points. Line integrals of conservative vector fields are independent of path.

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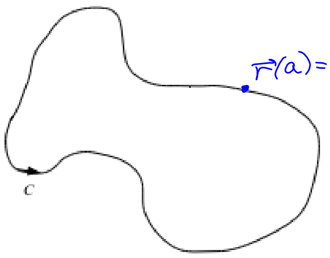
A curve is called **closed** if its terminal point coincides with its initial point, that is $\mathbf{r}(a) = \mathbf{r}(b)$.

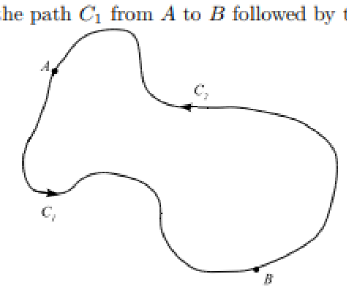


$\vec{r}(a) = \vec{r}(b)$

$\oint_C \vec{F} \cdot d\vec{r} = 0$

if $\vec{F}$ is conservative

If $\int_C \mathbf{F} \cdot d\mathbf{r}$ is independent of path in D and C is any closed path in D, we can choose any two points A and B on C and regard C as being composed of the path C_1 from A to B followed by the path C_2 from B to A.



Then

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \int_{C_1} \mathbf{F} \cdot d\mathbf{r} + \int_{C_2} \mathbf{F} \cdot d\mathbf{r} = \int_{C_1} \mathbf{F} \cdot d\mathbf{r} - \int_{-C_2} \mathbf{F} \cdot d\mathbf{r} = 0$$

Also we can show that if $\oint_C \mathbf{F} \cdot d\mathbf{r} = 0$ whenever C is a closed path in D, then $\int_C \mathbf{F} \cdot d\mathbf{r}$ is independent of path in D.

Theorem. $\int_C \mathbf{F} \cdot d\mathbf{r}$ is independent of path in D if and only if $\oint_C \mathbf{F} \cdot d\mathbf{r} = 0$ for every closed path in D.

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Now we assume that D is **open** (for every point P in D there is a disk with center P that lies entirely in D) and **connected** (any two points in D can be joined by a path that lies in D).

Theorem. Suppose $\mathbf{F}$ is a vector field that is continuous on an open connected region D. If $\int_C \mathbf{F}\cdot d\mathbf{r}$ is independent of path in D, then $\mathbf{F}$ is a conservative vector field on D; that is, there exists a function f such that $\nabla f = \mathbf{F}$.

Question: How to determine whether or not a vector field $\mathbf{F}$ is conservative?

Theorem. If $\mathbf{F}(x,y) = P(x,y)\mathbf{i} + Q(x,y)\mathbf{j}$ is a conservative vector fields, where P and Q have continuous first-order partial derivatives on a domain D, then throughout D we have

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

The converse of Theorem is true only for a special type of the region.

Definition. A curve is **simple** if it does not cross itself anywhere between its endpoints.

Definition. A **simply-connected region** in the plane is a connected region D such that every simple closed curve in D encloses only points that are in D (simply-connected region contains no hole and cannot consist of two separate pieces).

Theorem. Let $\mathbf{F} = P\mathbf{i} + Q\mathbf{j}$ be a vector field on an open simply-connected region D. Suppose that P and Q have continuous first-order derivatives and

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

Then $\mathbf{F}$ is conservative.

Example 1. Determine whether or not the vector field

$$\mathbf{F}(x,y) = (y\cos x - \cos y)\mathbf{i} + (\sin x + x\sin y)\mathbf{j}$$

is conservative.

$P(x,y) = y\cos x - \cos y$ | $Q(x,y) = \sin x + x\sin y$

$\frac{\partial P}{\partial y} = \cos x + \sin y$ $=$ $\frac{\partial Q}{\partial x} = \cos x + \sin y$

$\boxed{\text{YES}}$

Example 2.

1. If $\mathbf{F} = <2xy^3, 3x^2y^2>$, find a function f such that $\nabla f = \mathbf{F}$.

$P(x,y) = 2xy^3$ | $Q(x,y) = 3x^2y^2$ | $\vec{F}$ is conservative.

$\frac{\partial P}{\partial y} = 6xy^2 = \frac{\partial Q}{\partial x} = 6xy^2$

$\nabla f = < f_x, f_y > = < 2xy^3, 3x^2y^2 >$

$\int f_x dx = \int 2xy^3 dx \Rightarrow f(x,y) = \int 2xy^3 dx = x^2y^3 + g(y)$

$f_y = 3x^2y^2$

$\frac{\partial f}{\partial y} = 3x^2y^2 + g'(y)$

Plug into the 2nd equation:

$\cancel{3x^2y^2} + g'(y) = \cancel{3x^2y^2} \Rightarrow g'(y) = 0 \Rightarrow g(y) = C$

$\boxed{f(x,y) = x^2y^3 + \cancel{C}^{\,0}}$

2. Evaluate the line integral $\int_C \mathbf{F}\cdot d\mathbf{r}$ along the curve C given by $\mathbf{r}(t) = <\sin t, t^2+1>, 0 \le t \le \pi/2$.

$\int_C \vec{F}\cdot d\vec{r} = f(\vec{r}(\frac{\pi}{2})) - f(\vec{r}(0))$

$= f(1, \frac{\pi^2}{4}+1) - f(0,1)$

$= 1^2(\frac{\pi^2}{4}+1)^3 - 0^2 1^3 = \boxed{(\frac{\pi^2}{4}+1)^3}$

$\vec{r}(t) = < \sin t, t^2+1 >$

$\vec{r}(\frac{\pi}{2}) = < \sin\frac{\pi}{2}, \frac{\pi^2}{4}+1 > = < 1, \frac{\pi^2}{4}+1 >$

$\vec{r}(0) = < \sin 0, 0^2+1 > = < 0, 1 >$

9. Let $\vec{F}(x,y) = <\underbrace{2x + y^2 + 3x^2y}_{P(x,y)}, \underbrace{2xy + x^3 + 3y^2}_{Q(x,y)}>$. conservative indeed.

(a) Show that $\vec{F}$ is conservative vector field.

(b) Evaluate $\int_C \vec{F} \cdot d\vec{r}$ where C is the arc of the curve $y = x \sin x$ from $(0,0)$ to $(\pi, 0)$.

$$P(x,y) = 2x + y^2 + 3x^2 y \qquad Q(x,y) = 2xy + x^3 + 3y^2$$

$$\frac{\partial P}{\partial y} = 2y + 3x^2 = \frac{\partial Q}{\partial x} = 2y + 3x^2$$

Find a potential function $u = u(x,y)$ such that

$$\nabla u = <u_x, u_y> = \vec{F}(x,y) = <2x + y^2 + 3x^2 y, 2xy + x^3 + 3y^2>$$

$$\begin{cases} u_x = 2x + y^2 + 3x^2 y \\ \int u_y \, dy = \int [2xy + x^3 + 3y^2] \, dy \end{cases}$$

$$u(x,y) = xy^2 + x^3 y + y^3 + g(x)$$

$g(x)$ is an unkown function.

$$\frac{\partial u}{\partial x} = \cancel{y^2} + \cancel{3x^2 y} + 0 + g'(x) = 2x + \cancel{y^2} + \cancel{3x^2 y}$$

$$g'(x) = 2x \quad \text{or} \quad g(x) = x^2 + C$$

$$u(x,y) = xy^2 + x^3 y + y^3 + x^2 + \cancel{C}^{\,0}$$

$$\int_C \vec{F} \cdot d\vec{r} = u(\pi, 0) - u(0,0) = \pi(0^2) + \pi^3(0) + 0^3 + \pi^2 - 0 = \boxed{\pi^2}$$

Example 3.

1. If $\mathbf{F} = <2xz + \sin y, x \cos y, x^2>$, find a function f such that $\nabla f = \mathbf{F}$.

Find $u(x,y,z)$ such that

$$\nabla u = <u_x, u_y, u_z> = \vec{F} = <2xz + \sin y, x \cos y, x^2>$$

$$\begin{cases} \int u_x \, dx = \int [2xz + \sin y] \, dx \\ u_y = x \cos y \\ u_z = x^2 \end{cases} \Rightarrow u(x,y,z) = x^2 z + x \sin y + g(y,z)$$

g is an unknown function.

$$\frac{\partial u}{\partial y} = \cancel{x \cos y} + \frac{\partial g}{\partial y} = \cancel{x \cos y}$$

$$\frac{\partial g}{\partial y} = 0 \Rightarrow g(y,z) = g(z) \quad [g \text{ depends on } z \text{ only}].$$

$$u(x,y,z) = x^2 z + x \sin y + g(z)$$

$$\frac{\partial u}{\partial z} = \cancel{x^2} + g'(z) = \cancel{x^2} \Rightarrow g'(z) = 0 \Rightarrow g(z) = C \text{ constant.}$$

$$\boxed{u(x,y,z) = x^2 z + x \sin y + \cancel{C}^{\,0}}$$

2. Evaluate the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$ along the curve C given by $\mathbf{r}(t) = <\cos t, \sin t, t>$, $0 \le t \le 2\pi$.

$$\int_C \vec{F} \cdot d\vec{r} = u(\vec{r}(2\pi)) - u(\vec{r}(0))$$

$$\vec{r}(t) = <\cos t, \sin t, t> \Rightarrow \vec{r}(0) = <1, 0, 0>, \quad \vec{r}(2\pi) = <1, 0, 2\pi>$$

$$\int_C \vec{F} \cdot d\vec{r} = u(1, 0, 2\pi) - u(1,0,0) = 1^2(2\pi) + 1 \sin 0 - [1^2(0) + 1 \sin 0] = \boxed{2\pi}$$

Example 4. Show that the line integral $\int_C \underbrace{2x \sin y}^{P(x,y)}\, dx + \overbrace{(x^2 \cos y - 3y^2)}^{Q(x,y)}\, dy$ is independent of path and evaluate the integral if C is any path from (-1,0) to (5,1).

$P(x,y) = 2x \sin y$ | $Q(x,y) = x^2 \cos y - 3y^2$

$\frac{\partial P}{\partial y} = 2x \cos y \;=\; \frac{\partial Q}{\partial x} = 2x \cos y$

$\vec{F} = \langle 2x \sin y,\ x^2 \cos y - 3y^2 \rangle.$

Find its potential function. $u(x,y)$, such that

$\nabla u = \langle u_x, u_y \rangle = \langle 2x \sin y, x^2 \cos y - 3y^2 \rangle$

$$\begin{cases} u_x = 2x \sin y \\ \int u_y\, dy = \int [x^2 \cos y - 3y^2]\, dy \end{cases}$$

$u(x,y) = x^2 \sin y - y^3 + g(x)$, $g(x)$ is an unknown function

$\frac{\partial u}{\partial x} = 2x \sin y + g'(x) = 2x \sin y$

$g'(x) = 0$ or $g(x) = C$ constant

$u(x,y) = x^2 \sin y - y^3 + \cancel{C}^{\,0}$

$\int_C \vec{F} \cdot d\vec{r} = u(5,1) - u(-1,0) = \boxed{25 \sin 1 - 1}$