Section 16.4 **Green's Theorem.**

Definition. A **positively oriented curve** is a planar simple closed curve (that is, a curve in the plane whose starting point is also the end point and which has no other self-intersections) such that when traveling on it one always has the curve interior to the left (and consequently, the curve exterior to the right). If in the above definition one interchanges left and right, one obtains a **negatively oriented curve**.

Positive orientation   Negative orientation
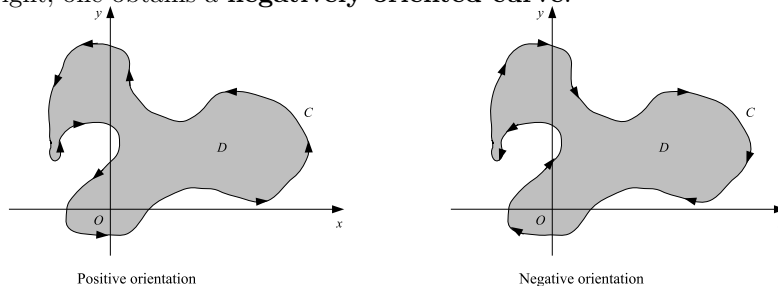
Green's Theorem. Let C be a positively oriented, piecewise-smooth, simple closed curve in the plane and let D be the region bounded by C. If P and Q have continuous partial derivatives on an open region that contains D, then

$$\oint_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

Note: Another notation for the positively oriented boundary curve of D is ∂D, so the equation in Green's Theorem can be written as
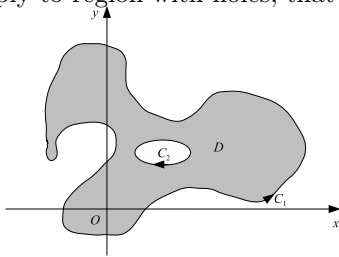
$$\oint_{\partial D} P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

Example 1. Use Green's Theorem to evaluate the integral

$$\oint_C (xy + e^{x^2})\,dx + (x^2 - \ln(1 + y))\,dy$$

if C consists of a line segment from $(0,0)$ to $(\pi, 0)$ and the curve $y = \sin x$, $0 \le x \le \pi$.

Green's Theorem can be extended to apply to region with holes, that is, regions that are not simply-connected.

Example 2. Use Green's Theorem to evaluate the integral

$$\oint_C (x^3 - y^3)dx + (x^3 + y^3)dy$$

if C is the boundary of the region between the circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 9$.

Application of Green's Theorem. The area of D is

$$A(D) = \oint_{\partial D} x\,dy = -\oint_{\partial D} y\,dx = \frac{1}{2}\oint_{\partial D} x\,dy - y\,dx$$

Example 3. Find the area of the region bounded by the curve with vector equation $\mathbf{r}(t) = \cos t\,\mathbf{i} + \sin^3 t\,\mathbf{j}$, $0 \le t \le 2\pi$.