Section 16.5 **Curl and divergence.**

Curl.

If $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$ is a vector field on $\mathbb{R}^3$ and the partial derivatives of P, Q, and R exist, then the **curl** of $\mathbf{F}$ is the vector field on $\mathbb{R}^3$ defined by

$$\text{curl } \mathbf{F} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\mathbf{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\mathbf{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\mathbf{k}$$

Let $\nabla = \frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}$ be the vector differential operator.

$$\nabla f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}$$

Then

$$\boxed{\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\mathbf{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\mathbf{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\mathbf{k} = \text{curl } \mathbf{F}}$$

Example 1. Find curl $\mathbf{F}$ if $\mathbf{F}(x, y, z) = xe^y\mathbf{i} - ze^{-y}\mathbf{j} + y\ln(z)\mathbf{k}$.

$$\text{curl } \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xe^y & -ze^{-y} & y\ln z \end{vmatrix} \begin{matrix} \vec{i} & \vec{j} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ xe^y & -ze^{-y} \end{matrix}$$

$$= \vec{i}\,\overset{\ln z}{\frac{\partial}{\partial y}(y\ln z)} + \vec{j}\,\overset{0}{\frac{\partial}{\partial z}(xe^y)} + \vec{k}\,\overset{0}{\frac{\partial}{\partial x}(-ze^{-y})}$$

$$- \vec{j}\,\overset{0}{\frac{\partial}{\partial x}(y\ln z)} - \vec{i}\,\overset{-e^{-y}}{\frac{\partial}{\partial z}(-ze^{-y})} - \vec{k}\,\overset{xe^y}{\frac{\partial}{\partial y}(xe^y)}$$

$$= \vec{i}\ln z + e^{-y}\vec{i} - \vec{k}\,xe^y = \boxed{\langle \ln z + e^{-y},\ 0,\ -xe^y \rangle}$$

Theorem 1. If f is a function of three variables that has continuous second-order partial derivatives, then

$$\text{curl}(\nabla f) = \mathbf{0}$$

Theorem 2. If $\mathbf{F}$ is a vector field defined on all on $\mathbb{R}^3$ whose component functions have continuous partial derivatives and curl $\mathbf{F} = \mathbf{0}$, then $\mathbf{F}$ is a conservative vector field.

Example 2. Determine whether or not the vector field $\mathbf{F} = zx\mathbf{i} + xy\mathbf{j} + yx\mathbf{k}$ is conservative. If it is conservative, find a function u such that $\mathbf{F} = \nabla u$.

$$\text{curl } \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ zx & xy & yx \end{vmatrix} \begin{matrix} \vec{i} & \vec{j} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ zx & xy \end{matrix}$$

$$= \vec{i}\,\overset{x}{\frac{\partial}{\partial y}(yx)} + \vec{j}\,\overset{x}{\frac{\partial}{\partial z}(zx)} + \vec{k}\,\overset{y}{\frac{\partial}{\partial x}(xy)}$$

$$- \vec{j}\,\overset{y}{\frac{\partial}{\partial x}(yx)} - \vec{i}\,\overset{0}{\frac{\partial}{\partial z}(xy)} - \vec{k}\,\overset{0}{\frac{\partial}{\partial y}(zx)}$$

$$= \vec{i}\,\vec{x} + \vec{j}(x-y) + \vec{k}\,\vec{y} \neq \langle 0, 0, 0 \rangle$$

$\boxed{\text{not conservative}}$

1. If $\mathbf{F} =< 2xz + \sin y, x\cos y, x^2 >$, find a function u such that $\nabla u = \mathbf{F}$.

Show that $\vec{F}$ is conservative.

$$\text{curl } \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xz+\sin y & x\cos y & x^2 \end{vmatrix} \begin{matrix} \vec{i} & \vec{j} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ 2xz+\sin y & x\cos y \end{matrix}$$

$$= \vec{i}\,\cancel{\frac{\partial}{\partial y}}(x^2)^{0} + \vec{j}\,\cancel{\frac{\partial}{\partial z}}(2xz+\sin y)^{2x} + \vec{k}\,\cancel{\frac{\partial}{\partial x}}(x\cos y)^{\cos y}$$

$$- \vec{j}\,\cancel{\frac{\partial}{\partial x}}(x^2)^{2x} - \vec{i}\,\cancel{\frac{\partial}{\partial z}}(x\cos y)^{0} - \vec{k}\,\cancel{\frac{\partial}{\partial y}}(2xz+\sin y)^{\cos y}$$

$$= \vec{i}\cdot 0 + \vec{j}\,(2x-2x) + \vec{k}\,(\cos y - \cos y) = <0,0,0>$$

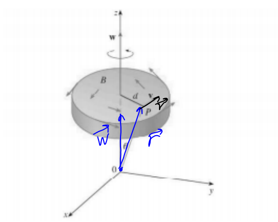
$\boxed{\text{conservative}}$

$$u = < u_x, u_y, u_z > = \vec{F} = < 2xz+\sin y, x\cos y, x^2 >$$

$$\begin{cases} u_x = 2xz + \sin y \\ u_y = x\cos y \\ u_z = x^2 \end{cases}$$

The curl vector is associated with rotation. If for a vector field $\mathbf{F}$ curl $\mathbf{F} = \mathbf{0}$, then the field $\mathbf{F}$ is irrotational.

Example 3. Let B be a rigid body rotation about the z-axis. The rotation can be described by the vector $\mathbf{w} = \omega\mathbf{k}$, where ω is the angular speed of B, that is the tangential speed of any point P in B divided by the distance d from the axis of rotation. Let $\mathbf{r} =< x, y, z >$ be the position vector of P.



1. Show that velocity field of B is given by $\mathbf{v} = \mathbf{w} \times \mathbf{r}$

$\vec{v}$ is perpendicular to both $\vec{w}$ and $\vec{r}$

$|\vec{v}| =$

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2. Show that $\mathbf{v} = -\omega y\mathbf{i} + \omega x\mathbf{j}$

3. Show that curl $\mathbf{v} = 2\mathbf{w}$.

Divergence.

Definition. If $\mathbf{F} =< P, Q, R >$ is a vector field in $\mathbb{R}^3$ and P_x, Q_y, and R_z exist, then the **divergence** of $\mathbf{F}$ is the function of tree variables defined by

$$\boxed{\text{div } \mathbf{F} = \nabla \cdot \mathbf{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}}$$

Example 4. Find the divergence of the vector field $\mathbf{F}(x, y, z) = xe^y\mathbf{i} - ze^{-y}\mathbf{j} + y\ln(z)\mathbf{k}$.

$$\text{div } \vec{F} = \frac{\partial}{\partial x}(xe^y) + \frac{\partial}{\partial y}(-ze^{-y}) + \frac{\partial}{\partial z}(y\ln z)$$

$$= \boxed{e^y + ze^{-y} + \frac{y}{z}}$$

Theorem 3. If $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + Q\mathbf{k}$ is a vector field on $\mathbb{R}^3$ and P, Q, and R have continuous second-order derivatives, then

$$\text{div curl } \mathbf{F} = 0$$

Divergence is a vector operator that measures the magnitude of a vector field's source or sink at a given point, in terms of a signed scalar. If div $\mathbf{F} = 0$, then $\mathbf{F}$ is said to be **incompressible**.

Laplace operator:

$$\text{div}(\nabla f) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = \nabla^2 f$$

Properties of the curl and divergence.

If f is a scalar field and $\mathbf{F}$, $\mathbf{G}$ are vector fields, then $f\mathbf{F}$, $\mathbf{F} \cdot \mathbf{G}$, and $\mathbf{F} \times \mathbf{G}$ are vector fields defined by

$$(f\mathbf{F})(x, y, z) = f(x, y, z)\mathbf{F}(x, y, z)$$

$$(\mathbf{F} \cdot \mathbf{G})(x, y, z) = \mathbf{F}(x, y, z) \cdot \mathbf{G}(x, y, z)$$

$$(\mathbf{F} \times \mathbf{G})(x, y, z) = \mathbf{F}(x, y, z) \times \mathbf{G}(x, y, z)$$

and

1. $\text{div}(\mathbf{F} + \mathbf{G}) = \text{div } \mathbf{F} + \text{div } \mathbf{G}$
2. $\text{curl}(\mathbf{F} + \mathbf{G}) = \text{curl } \mathbf{F} + \text{curl } \mathbf{G}$
3. $\text{div}(f\mathbf{F}) = f\text{div } \mathbf{F} + \mathbf{F} \cdot \nabla f$
4. $\text{curl}(f\mathbf{F}) = f\text{curl } \mathbf{F} + (\nabla f) \times \mathbf{F}$
5. $\text{div}(\mathbf{F} \times \mathbf{G}) = \mathbf{G} \cdot \text{curl } \mathbf{F} - \mathbf{F} \cdot \text{curl } \mathbf{G}$
6. $\text{div}(\nabla f \times \nabla g) = 0$
7. $\text{curl curl}(\mathbf{F}) = \text{grad div } \mathbf{F} - \nabla^2\mathbf{F}$
8. $\nabla(\mathbf{F} \cdot \mathbf{G}) = (\mathbf{F} \cdot \nabla)\mathbf{G} + (\mathbf{G} \cdot \nabla)\mathbf{F} + \mathbf{F} \times \text{curl } \mathbf{G} + \mathbf{G} \times \text{curl } \mathbf{F}$

D C

$$\oint_C P\,dx + Q\,dy = \iint_D \left[\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right] dA$$

Vector forms of Green's Theorem.

Let $\mathbf{F} = P\mathbf{i} + Q\mathbf{j}$ be a vector field. We suppose that the plane region D, its boundary curve C, and the functio P and Q satisfy the hypotheses of Green's Theorem. Then

$$\oint_C \mathbf{F}\cdot d\mathbf{r} = \oint_C P dx + Q dy$$

and

$$\text{curl } \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ P(x,y) & Q(x,y) & 0 \end{vmatrix} = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\mathbf{k}$$

Therefore $$(\text{curl } \mathbf{F})\cdot\mathbf{k} = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\mathbf{k}\cdot\mathbf{k} = \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$$

and we can rewrite the equation in Green's Theorem in the vector form

$$\oint_C \mathbf{F}\cdot d\mathbf{r} = \iint_D (\text{curl } \mathbf{F})\cdot\mathbf{k}\, dA$$

If C is given by the vector equation $\mathbf{r}(t) =< x(t), y(t) >$, $a \le t \le b$, then the unit tangent vector

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \frac{x'(t)}{|\mathbf{r}'(t)|}\mathbf{i} + \frac{y'(t)}{|\mathbf{r}'(t)|}\mathbf{j}$$

Then the outward unit normal vector to C is given by

$$\mathbf{n}(t) = \frac{y'(t)}{|\mathbf{r}'(t)|}\mathbf{i} - \frac{x'(t)}{|\mathbf{r}'(t)|}\mathbf{j}$$

Then

$$\oint_C \mathbf{F}\cdot\mathbf{n}\, ds = \int_a^b (\mathbf{F}\cdot\mathbf{n})(t)|\mathbf{r}'(t)|dt$$

$$= \int_C P dy - Q dx = \iint \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\right) dA$$

So,

$$\boxed{\oint_C \mathbf{F}\cdot\mathbf{n} ds = \iint_D \text{div } \mathbf{F}(x,y) dA}$$