

## Section 16.6 Parametric surfaces and their areas.

We suppose that

$$\mathbf{r}(u, v) = x(u, v)\mathbf{i} + y(u, v)\mathbf{j} + z(u, v)\mathbf{k}$$

is a vector-valued function defined on an region  $D$  in the  $uv$ -plane and the partial derivatives of  $x$ ,  $y$ , and  $z$  with respect to  $u$  and  $v$  are all continuous. The set of all points  $(x, y, z) \in \mathbb{R}^3$ , such that

$$x = x(u, v), \quad y = y(u, v) \quad z = z(u, v)$$

and  $(u, v) \in D$ , is called a **parametric surface**  $S$  with parametric equations

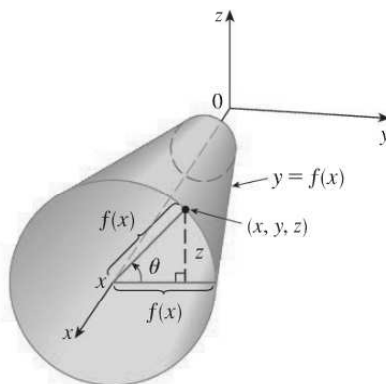
$$x = x(u, v), \quad y = y(u, v) \quad z = z(u, v).$$

**Example 1.** Find a parametric representation for the part of the elliptic paraboloid  $y = 6 - 3x^2 - 2z^2$  that lies to the right of the  $xz$ -plane.

In general, a surface given as the graph of the function  $z = f(x, y)$ , can always be regarded as a parametric surface with parametric equations

$$x = x, \quad y = y \quad z = f(x, y).$$

Surfaces of revolution also can be represented parametrically. Let us consider the surface  $S$  obtained by rotating the curve  $y = f(x)$ ,  $a \leq x \leq b$ , about the  $x$ -axis, where  $f(x) \geq 0$  and  $f'$  is continuous.



Let  $\theta$  be the angle of rotation. If  $(x, y, z)$  is a point on  $S$ , then

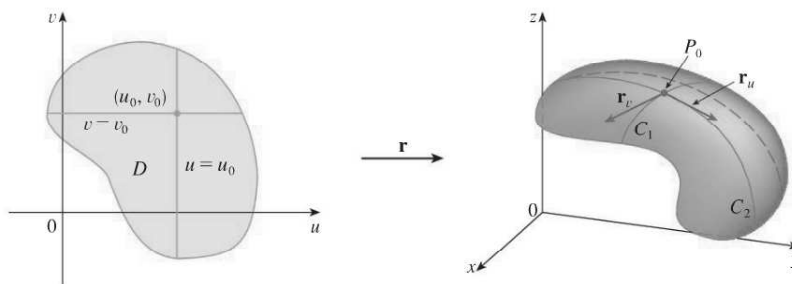
$$\boxed{x = x} \quad \boxed{y = f(x) \cos \theta} \quad \boxed{z = f(x) \sin \theta}$$

The parameter domain is given by  $a \leq x \leq b$ ,  $0 \leq \theta \leq 2\pi$ .

**Example 2.** Find equation for the surface generated by rotating the curve  $x = 4y^2 - y^4$ ,  $-2 \leq y \leq 2$ , about the  $y$ -axis.

### Tangent planes.

**Problem.** Find the tangent plane to a parametric surface  $S$  given by a vector function  $\mathbf{r}(u, v)$  at a point  $P_0$  with position vector  $\mathbf{r}(u_0, v_0)$ .



The tangent vector  $\mathbf{r}_v$  to  $C_1$  at  $P_0$  is

$$\mathbf{r}_v = \frac{\partial x}{\partial v}(u_0, v_0)\mathbf{i} + \frac{\partial y}{\partial v}(u_0, v_0)\mathbf{j} + \frac{\partial z}{\partial v}(u_0, v_0)\mathbf{k}$$

Similarly, the tangent vector  $\mathbf{r}_u$  to  $C_2$  at  $P_0$  is

$$\mathbf{r}_u = \frac{\partial x}{\partial u}(u_0, v_0)\mathbf{i} + \frac{\partial y}{\partial u}(u_0, v_0)\mathbf{j} + \frac{\partial z}{\partial u}(u_0, v_0)\mathbf{k}$$

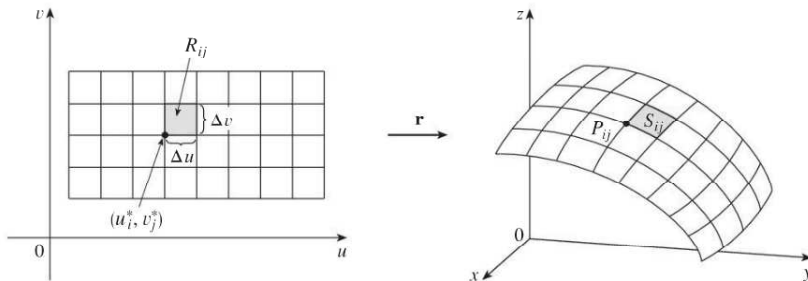
Then the **normal vector** to the tangent plane to a parametric surface  $S$  at  $P_0$  is the vector

$$\boxed{\mathbf{n} = \mathbf{r}_u \times \mathbf{r}_v}$$

If  $\mathbf{r}_u \times \mathbf{r}_v \neq \vec{0}$ , then  $S$  is called **smooth**.

**Example 3.** Find the tangent plane to the surface with parametric equations  $\mathbf{r}(u, v) = (u+v)\mathbf{i} + u \cos v\mathbf{j} + v \sin v\mathbf{k}$  at the point  $(1, 1, 0)$ .

**Surface area.** Let  $S$  be a parametric surface given by a vector function  $\mathbf{r}(u, v)$ ,  $(u, v) \in D$ . For simplicity, we start by considering a surface whose parameter domain  $D$  is a rectangle, and we partition it into subrectangles  $R_{ij}$ .



Let's choose  $(u_i^*, v_j^*)$  to be the lower left corner of  $R_{ij}$ . The part  $S_{ij}$  of the surface  $S$  that corresponds to  $R_{ij}$  has the point  $P_{ij}$  with position vector  $\mathbf{r}(u_i^*, v_j^*)$  as one of its corners. Let

$$\mathbf{r}_{u_i} = \mathbf{r}_u(u_i^*, v_j^*), \quad \mathbf{r}_{v_j} = \mathbf{r}_v(u_i^*, v_j^*)$$

be the tangent vectors at  $P_{ij}$ . We approximate  $S_{ij}$  by the parallelogram determined by the vectors  $\Delta u_i \mathbf{r}_{u_i}$  and  $\Delta v_j \mathbf{r}_{v_j}$  (this parallelogram lies in the tangent plane to  $S$  at  $P_{ij}$ ). The area of this parallelogram is

$$|(\Delta u_i \mathbf{r}_{u_i}) \times (\Delta v_j \mathbf{r}_{v_j})| = |\mathbf{r}_{u_i} \times \mathbf{r}_{v_j}| \Delta u_i \Delta v_j$$

so an approximation to the area of  $S$  is

$$\sum_{i=1}^m \sum_{j=1}^n |\mathbf{r}_{u_i} \times \mathbf{r}_{v_j}| \Delta u_i \Delta v_j \rightarrow \iint_D |\mathbf{r}_u \times \mathbf{r}_v| dA \text{ as } \|P\| \rightarrow 0.$$

**Definition.** If a smooth parametric surface  $S$  is given by the equation  $\mathbf{r}(u, v) = x(u, v)\mathbf{i} + y(u, v)\mathbf{j} + z(u, v)\mathbf{k}$ ,  $(u, v) \in D$  and  $S$  is covered just once at  $(u, v)$  ranges throughout the parameter domain  $D$ , then the **surface area** of  $S$  is

$$A(S) = \iint_D |\mathbf{r}_u \times \mathbf{r}_v| dA$$

where

$$\mathbf{r}_u = \frac{\partial x}{\partial u} \mathbf{i} + \frac{\partial y}{\partial u} \mathbf{j} + \frac{\partial z}{\partial u} \mathbf{k} \quad \mathbf{r}_v = \frac{\partial x}{\partial v} \mathbf{i} + \frac{\partial y}{\partial v} \mathbf{j} + \frac{\partial z}{\partial v} \mathbf{k}$$

If a surface  $S$  is given by  $z = f(x, y)$ ,  $(x, y) \in D$ , the parametric equations for  $S$  are

$$x = x, \quad y = y, \quad z = z(x, y)$$

Then  $\mathbf{r}_x = \langle 1, 0, z_x(x, y) \rangle$ ,  $\mathbf{r}_y = \langle 0, 1, z_y(x, y) \rangle$ , and

$$\mathbf{r}_x \times \mathbf{r}_y = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & z_x(x, y) \\ 0 & 1 & z_y(x, y) \end{vmatrix} = -z_x \mathbf{i} - z_y \mathbf{j} + \mathbf{k}$$

Then

$$|\mathbf{r}_x \times \mathbf{r}_y| = \sqrt{1 + [z_x(x, y)]^2 + [z_y(x, y)]^2}$$

and

$$A(S) = \iint_D \sqrt{1 + [z_x(x, y)]^2 + [z_y(x, y)]^2} dA$$

**Example 4.** Find the surface area of the part of the surface  $z = x + y^2$  that lies above the triangle with vertices  $(0,0)$ ,  $(1,1)$ , and  $(0,1)$ .

**Example 5.** Find the surface area of the part of the sphere  $x^2 + y^2 + z^2 = a^2$  that lies inside the cylinder  $x^2 + y^2 = ax$ .