Section 16.6 **Parametric surfaces and their areas.**

We suppose that

$$\mathbf{r}(u,v) = x(u,v)\mathbf{i} + y(u,v)\mathbf{j} + z(u,v)\mathbf{k}$$

is a vector-valued function defined on an region D in the uv-plane and the partial derivatives of x, y, and z with respect to u and v are all continuous. The set of all points $(x,y,z) \in \mathbb{R}^3$, such that

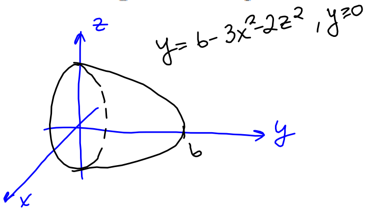
$$x = x(u,v), \quad y = y(u,v) \quad z = z(u,v)$$

and $(u,v) \in D$, is called a parametric surface S with parametric equations

$$\boxed{x = x(u,v),} \quad \boxed{y = y(u,v)} \quad \boxed{z = z(u,v).}$$

The region D is called the parameter domain.

Example 1. Find a parametric representation for the part of the elliptic paraboloid $y = 6 - 3x^2 - 2z^2$ that lies to the right of the xz-plane.



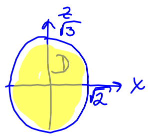
$$\begin{cases} x = x \\ y = 6 - 3x^2 - 2z^2 \\ z = z \end{cases}$$

Parameter domain:

$6 - 3x^2 - 2z^2 \geq 0$

$3x^2 + 2z^2 \leq 6$

$\frac{x^2}{2} + \frac{z^2}{3} \leq 1$



$x = \sqrt{2}\, r\cos\theta$

$z = \sqrt{3}\, r\sin\theta$

$y = 6 - 3[\sqrt{2}\, r\cos\theta]^2 - 2[\sqrt{3}\, r\sin\theta]^2$

$= 6 - 6r^2\cos^2\theta - 6r^2\sin^2\theta = 6 - 6r^2\overbrace{(\cos^2\theta + \sin^2\theta)}^{1}$

$y = 6 - 6r^2$

$$\begin{cases} x = \sqrt{2}\, r\cos\theta \\ y = 6 - 6r^2 \\ z = \sqrt{3}\, r\sin\theta \end{cases},$$

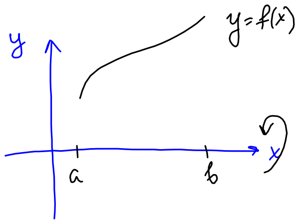
parameter domain $\boxed{0 \leq \theta \leq 2\pi}$

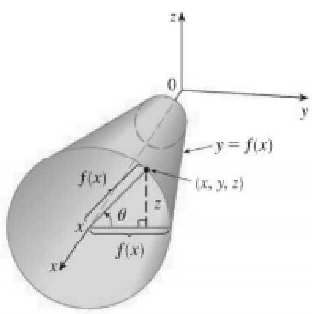
$y = 6 - 6r^2$, $y \geq 0 \Longrightarrow 6 - 6r^2 \geq 0 \Rightarrow r^2 \leq 1$ $\boxed{0 \leq r \leq 1}$

In general, a surface given as the graph of the function $z = f(x,y)$, can always be regarded as a parametric surface with parametric equations

$$\boxed{x = x,} \quad \boxed{y = y} \quad \boxed{z = f(x,y).}$$

Surfaces of revolution also can be represented parametrically. Let us consider the surface S obtained by rotating the curve $y = f(x)$, $a \leq x \leq b$, about the x-axis, where $f(x) \geq 0$ and f' is continuous.





Let θ be the angle of rotation. If (x,y,z) is a point on S, then

$$\boxed{x = x} \quad \boxed{y = f(x)\cos\theta} \quad \boxed{z = f(x)\sin\theta}$$

The parameter domain is given by $a \leq x \leq b$, $0 \leq \theta \leq 2\pi$.

Example 2. Find equation for the surface generated by rotating the curve $x = 4y^2 - y^4$, $-2 \leq y \leq 2$, about the y-axis. θ is the angle of rotation.

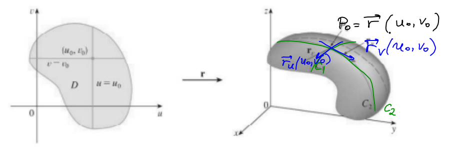
$$\begin{cases} y = y \\ x = (4y^2 - y^4)\cos\theta \\ z = (4y^2 - y^4)\sin\theta \end{cases}$$

Parameter domain $0 \leq \theta \leq 2\pi$

$-2 \leq y \leq 2$

Tangent planes.

Problem. Find the tangent plane to a parametric surface S given by a vector function $\mathbf{r}(u,v)$ at a point P_0 with position vector $\mathbf{r}(u_0,v_0)$.



The tangent vector $\mathbf{r}_v$ to C_1 at P_0 is

$$\mathbf{r}_v = \frac{\partial x}{\partial v}(u_0,v_0)\mathbf{i} + \frac{\partial y}{\partial v}(u_0,v_0)\mathbf{j} + \frac{\partial z}{\partial v}(u_0,v_0)\mathbf{k}$$

Similarly, the tangent vector $\mathbf{r}_u$ to C_2 at P_0 is

$$\mathbf{r}_u = \frac{\partial x}{\partial u}(u_0,v_0)\mathbf{i} + \frac{\partial y}{\partial u}(u_0,v_0)\mathbf{j} + \frac{\partial z}{\partial u}(u_0,v_0)\mathbf{k}$$

Then the normal vector to the tangent plane to a parametric surface S at P_0 is the vector

$$\boxed{\mathbf{n} = \mathbf{r}_u \times \mathbf{r}_v}$$

If $\mathbf{r}_u \times \mathbf{r}_v \neq \vec{0}$, then S is called **smooth**.

Example 3. Find the tangent plane to the surface with parametric equations $\mathbf{r}(u,v) = (u+v)\mathbf{i} + u\cos v\mathbf{j} + v\sin u\mathbf{k}$ at the point $(1,1,0)$.

$\vec{r}(u,v) = \langle u+v,\ u\cos v,\ v\sin u \rangle$

Find u_0, v_0 such that $\vec{r}(u_0,v_0) = \langle 1,1,0 \rangle$ | $\vec{r}(1,0) = \langle 1,1,0 \rangle$

$$\begin{cases} u_0 + v_0 = 1 \\ u_0\cos v_0 = 1 \\ v_0 \sin u_0 = 0 \end{cases} \qquad v_0 = 0,\ u_0 = 1$$

$\vec{r}_u(u,v) = \langle 1, \cos v, v\cos u \rangle$ | $\vec{r}_v(u,v) = \langle 1, -u\sin v, \sin u \rangle$

$\vec{r}_u(1,0) = \langle 1, \cos 0, 0\cos 1 \rangle = \langle 1,1,0 \rangle$ | $\vec{r}_v(1,0) = \langle 1, -1\sin 0, \sin 1 \rangle = \langle 1, 0, \sin 1 \rangle$

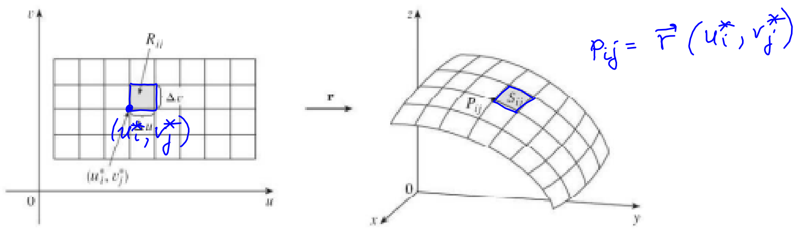
Normal vector to the tangent plane:

$$\vec{n} = \vec{r}_u(1,0) \times \vec{r}_v(1,0) = \langle 1,1,0 \rangle \times \langle 1,0,\sin 1 \rangle = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 1 & 0 \\ 1 & 0 & \sin 1 \end{vmatrix}$$

$$= \sin 1\,\vec{i} - \vec{k} - \sin 1\,\vec{j} = \langle \sin 1, -\sin 1, -1 \rangle$$

Tangent plane: $\boxed{(\sin 1)(x-1) - (\sin 1)(y-1) - 1(z-0) = 0}$

Surface area. Let S be a parametric surface given by a vector function $\mathbf{r}(u,v)$, $(u,v) \in D$. For simplicity, we start by considering a surface whose parameter domain D is a rectangle, and we partition it into subrectangles R_{ij}.



Let's choose (u_i^*, v_j^*) to be the lower left corner of R_{ij}. The part S_{ij} of the surface S that corresponds to R_{ij} has the point P_{ij} with position vector $\mathbf{r}(u_i^*, v_j^*)$ as one of its corners. Let

$$\mathbf{r}_{u_i} = \mathbf{r}_u(u_i^*, v_j^*), \qquad \mathbf{r}_{v_j} = \mathbf{r}_v(u_i^*, v_j^*)$$

be the tangent vectors at P_{ij}. We approximate S_{ij} by the parallelogram determined by the vectors $\Delta u_i \mathbf{r}_{u_i}$ and $\Delta v_j \mathbf{r}_{v_j}$ (this parallelogram lies in the tangent plane to S at P_{ij}). The area of this parallelogram is

$$|(\Delta u_i \mathbf{r}_{u_i}) \times (\Delta v_j \mathbf{r}_{v_j})| = |\mathbf{r}_{u_i} \times \mathbf{r}_{v_j}|\Delta u_i \Delta v_j$$

so an approximation to the area of S is

$$\sum_{i=1}^{m}\sum_{j=1}^{n} |\mathbf{r}_{u_i} \times \mathbf{r}_{v_j}|\Delta u_i \Delta v_j \to \iint_D |\mathbf{r}_u \times \mathbf{r}_v|\,dA \text{ as } \|P\| \to 0.$$

Definition. If a smooth parametric surface S is given by the equation $\mathbf{r}(u,v) = x(u,v)\mathbf{i} + y(u,v)\mathbf{j} + z(u,v)\mathbf{k}$, $(u,v) \in D$ and S is covered just once at (u,v) ranges throughout the parameter domain D, then the **surface area** of S is

$$A(S) = \iint_D \underbrace{|\mathbf{r}_u \times \mathbf{r}_v|}_{\text{magnitude of the normal vector}} dA$$

where

$$\mathbf{r}_u = \frac{\partial x}{\partial u}\mathbf{i} + \frac{\partial y}{\partial u}\mathbf{j} + \frac{\partial z}{\partial u}\mathbf{k} \qquad \mathbf{r}_v = \frac{\partial x}{\partial v}\mathbf{i} + \frac{\partial y}{\partial v}\mathbf{j} + \frac{\partial z}{\partial v}\mathbf{k}$$

If a surface S is given by $z = z(x,y)$, $(x,y) \in D$, the parametric equations for S are

$$x = x, \qquad y = y \qquad z = z(x,y)$$

Then $\mathbf{r}_x = <1, 0, z_x(x,y)>$, $\mathbf{r}_y = <0, 1, z_y(x,y)>$, and

$$\mathbf{r}_x \times \mathbf{r}_y = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & z_x(x,y) \\ 0 & 1 & z_y(x,y) \end{vmatrix} = -z_x\mathbf{i} - z_y\mathbf{j} + \mathbf{k}$$

Then

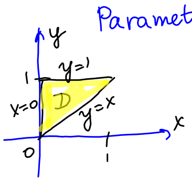
$$|\mathbf{r}_x \times \mathbf{r}_y| = \sqrt{1 + [z_x(x,y)]^2 + [z_y(x,y)]^2}$$

and

$$\boxed{A(S) = \iint_D \sqrt{1 + [z_x(x,y)]^2 + [z_y(x,y)]^2}\, dA}$$

Example 4. Find the surface area of the part of the surface $z = x + y^2$ that lies above the triangle with vertices (0,0), (1,1), and (0,1).

Parameter domain:



$0 \le x \le y$
$0 \le y \le 1$

$$A(S) = \iint_D \sqrt{1 + [z_x]^2 + [z_y]^2}\, dA$$

$$z_x = 1, \quad z_y = 2y$$

$$A(S) = \iint_D \sqrt{1 + 1^2 + (2y)^2}\, dA = \iint_D \sqrt{2 + 4y^2}\, dA$$

$$= \int_0^1 \int_0^y \sqrt{2+4y^2}\, dx\, dy = \int_0^1 \sqrt{2+4y^2}\, x \Big|_0^y dy = \int_0^1 y\sqrt{2+4y^2}\, dy \quad \left| \begin{array}{l} u = 2+4y^2 \\ du = 8y\, dy \\ 2 \le u \le 6 \end{array} \right|$$

$$= \frac{1}{8}\int_2^6 \sqrt{u}\, du = \frac{1}{8} \frac{u^{3/2}}{3/2}\Big|_2^6 = \boxed{\frac{1}{8} \cdot \frac{2}{3}\left(6^{3/2} - 2^{3/2}\right)}$$

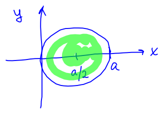
Example 5. Find the surface area of the part of the sphere $x^2+y^2+z^2=a^2$ that lies inside the cylinder $x^2+y^2=ax$, $a>0$.

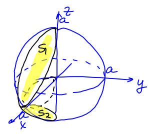
$x^2-\underbrace{ax}_{2}+y^2=0$

Complete the square:

$$x^2-2\cdot\frac{a}{2}x+\left(\frac{a}{2}\right)^2-\left(\frac{a}{2}\right)^2+y^2=0$$

$$\left(x-\frac{a}{2}\right)^2+y^2=\frac{a^2}{4}$$





$0\le\varphi\le\frac{\pi}{2}$

$-\frac{\pi}{2}\le\theta\le\frac{\pi}{2}$

$$A(S)=A(S_1)+A(S_2)=2A(S_1)$$

Paramertization of the sphere:

$$\begin{cases} x=a\cos\theta\sin\varphi \\ y=a\sin\theta\sin\varphi \\ z=a\cos\varphi \end{cases}$$

$$\vec{r}(\theta,\varphi)=\langle a\cos\theta\sin\varphi,\ a\sin\theta\sin\varphi,\ a\cos\varphi\rangle$$

$$\vec{r}_\theta=\langle -a\sin\theta\sin\varphi,\ a\cos\theta\sin\varphi,\ 0\rangle$$

$$\vec{r}_\varphi=\langle a\cos\theta\cos\varphi,\ a\sin\theta\cos\varphi,\ -a\sin\varphi\rangle$$

$$\vec{r}_\theta\times\vec{r}_\varphi=\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -a\sin\theta\sin\varphi & a\cos\theta\sin\varphi & 0 \\ a\cos\theta\cos\varphi & a\sin\theta\cos\varphi & -a\sin\varphi \end{vmatrix} = \vec{i}\left(-a^2\cos\theta\sin^2\varphi-0\right)-\vec{j}\left(a^2\sin\theta\sin^2\varphi-0\right)+\vec{k}\left(-a^2\sin^2\theta\sin\varphi\cos\varphi-a^2\cos^2\theta\cos\varphi\sin\varphi\right)$$

$$=-a^2\cos\theta\sin^2\varphi\,\vec{i}-a^2\sin\theta\sin^2\varphi\,\vec{j}-a^2\sin\varphi\cos\varphi\underbrace{\left(\sin^2\theta+\cos^2\theta\right)}_{1}\vec{k}$$

$$=\langle -a^2\cos\theta\sin^2\varphi,\ -a^2\sin\theta\sin^2\varphi,\ -a^2\sin\varphi\cos\varphi\rangle$$

$$|\vec{r}_\theta\times\vec{r}_\varphi|=\sqrt{a^4\cos^2\theta\sin^4\varphi+a^4\sin^2\theta\sin^4\varphi+a^4\sin^2\varphi\cos^2\varphi}$$

$$=\sqrt{a^4\sin^2\varphi\left[\left(\cos^2\theta\sin^2\varphi+\sin^2\theta\sin^2\varphi\right)+\cos^2\varphi\right]}=a^2\sin\varphi\sqrt{\sin^2\varphi\underbrace{\left(\cos^2\theta+\sin^2\theta\right)}_{1}+\cos^2\varphi}$$

$$=a^2\sin\varphi\sqrt{\underbrace{\cos^2\varphi+\sin^2\varphi}_{1}}=a^2\sin\varphi$$

$$A(S_1)=\iint_D|\vec{r}_\theta\times\vec{r}_\varphi|\,d\varphi\,d\theta=a^2\int_{-\pi/2}^{\pi/2}\int_0^{\pi/2}\sin\varphi\,d\varphi\,d\theta$$

$$=a^2\int_{-\pi/2}^{\pi/2}\left[-\cos\varphi\right]_0^{\pi/2}d\theta=a^2\int_{-\pi/2}^{\pi/2}\left[-\overset{0}{\cancel{\cos\frac{\pi}{2}}}+\overset{1}{\cancel{\cos 0}}\right]d\theta=a^2\int_{-\pi/2}^{\pi/2}d\theta=a^2\pi$$

$$A(S)=2A(S_1)=\boxed{2a^2\pi}$$