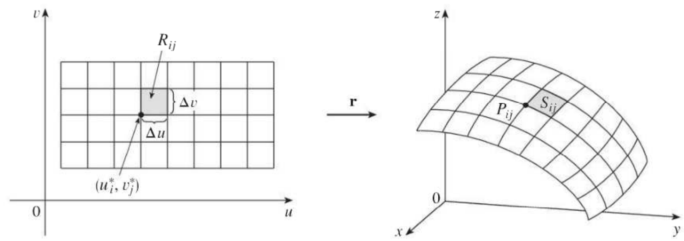
Section 16.7 **Surface integrals.**

Suppose f is a function of three variables whose domain include a surface S.



We divide S into patches S_{ij} with area ΔS_{ij}. We evaluate f at a point P_{ij}^* in each patch, multiply by the area ΔS_{ij}, and form the sum

$$\sum_{i=1}^{m}\sum_{j=1}^{n} f(P_{ij}^*)\Delta S_{ij}$$

We define the **surface integral of f over the surface S** as

$$\iint_S f(x,y,z)dS = \lim_{\|P\|\to 0}\sum_{i=1}^{m}\sum_{j=1}^{n} f(P_{ij}^*)\Delta S_{ij}$$

If the surface S is given by an equation $z = z(x,y)$, $(x,y) \in D$, then

$$\iint_S f(x,y,z)dS = \iint_D f(x,y,z(x,y))\sqrt{\left[\frac{\partial z}{\partial x}\right]^2 + \left[\frac{\partial z}{\partial y}\right]^2 + 1}\,dA$$

If the surface S is given by vector function $\mathbf{r}(u,v) = x(u,v)\mathbf{i} + y(u,v)\mathbf{j} + z(u,v)\mathbf{k}$, $(u,v) \in D$, then

$$\iint_S f(x,y,z)dS = \iint_D f(\mathbf{r}(u,v))|\mathbf{r}_u \times \mathbf{r}_v|\,dA$$

where

$$\mathbf{r}_u = \frac{\partial x}{\partial u}\mathbf{i} + \frac{\partial y}{\partial u}\mathbf{j} + \frac{\partial z}{\partial u}\mathbf{k} \qquad \mathbf{r}_v = \frac{\partial x}{\partial v}\mathbf{i} + \frac{\partial y}{\partial v}\mathbf{j} + \frac{\partial z}{\partial v}\mathbf{k}$$

Example 1.

1. Evaluate $\iint_S y\, dS$, where S is the part of the plane $3x + 2y + z = 6$ that lies in the first octant.

2. Evaluate $\iint_S \sqrt{1+x^2+y^2}\, dS$, if S is given by vector equation $\mathbf{r}(u,v) = u\cos v\mathbf{i} + u\sin v\mathbf{j} + v\mathbf{k}$, $0 \le u \le 1$, $0 \le v \le \pi$.

3. Evaluate $\iint_S xy\,dS$, if S is a boundary of the region enclosed by the cylinder $x^2 + z^2 = 1$ and the planes $y = 0$ and $x + y = 2$.

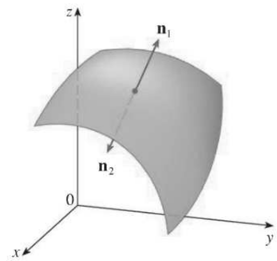
If a thin sheet has the shape of a surface S and the density at the point (x, y, z) is $\rho(x, y, z)$, then the **total mass** of the sheet is

$$\boxed{m = \iint\limits_S \rho(x, y, z)dS}$$

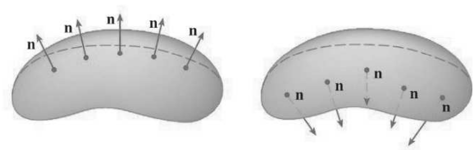
and the **center of mass** is $(\bar{x}, \bar{y}, \bar{z})$, where

$$\boxed{\bar{x} = \frac{1}{m}\iint\limits_S x\rho(x, y, z)dS} \quad \boxed{\bar{y} = \frac{1}{m}\iint\limits_S y\rho(x, y, z)dS} \quad \boxed{\bar{z} = \frac{1}{m}\iint\limits_S z\rho(x, y, z)dS}$$

Oriented surfaces.
Let us consider a surface S that has a tangent plane at every point (x, y, z) on S (except at any boundary point).



There are two unit normal vectors $\mathbf{n}_1$ and $\mathbf{n}_2 = -\mathbf{n}_1$ at (x, y, z). If it is possible to chose a unit normal vector $\mathbf{n}$ at every such point (x, y, z) so that $\mathbf{n}$ varies continuously over S, then S is called an **oriented surface** and the given choice of $\mathbf{n}$ provides S with an **orientation**. There are two possible orientations for any orientable surface.



For a surface $z = z(x, y)$ the orientation is given by the unit normal vector

$$\boxed{\mathbf{n} = \frac{-\dfrac{\partial z}{\partial x}\mathbf{i} - \dfrac{\partial z}{\partial y}\mathbf{j} + \mathbf{k}}{\sqrt{\left[\dfrac{\partial z}{\partial x}\right]^2 + \left[\dfrac{\partial z}{\partial y}\right]^2 + 1}}}$$

Since the $\mathbf{k}$-component is positive, this gives the *upward* orientation of the surface.

If S is a smooth orientable surface given in parametric form by a vector function $\mathbf{r}(u, v)$, then its orientation is given by a unit normal vector

$$\boxed{\mathbf{n} = \frac{\mathbf{r}_u \times \mathbf{r}_v}{|\mathbf{r}_u \times \mathbf{r}_v|}}$$

For a **closed surface**, the positive orientation is the one for which the normal vectors point *outward* from S, the inward-pointing normals give the negative orientation.

Surface integrals of vector fields.

Definition. If $\mathbf{F}$ is a continuous vector-field defined on an oriented surface S with normal vector $\mathbf{n}$, then the **surface integral of F over S** is

$$\boxed{\iint\limits_S \mathbf{F}\cdot d\mathbf{S} = \iint\limits_S \mathbf{F}\cdot\mathbf{n}\,dS}$$

This integral is also called the **flux** of $\mathbf{F}$ across S.

If $\mathbf{F} =< P(x,y,z), Q(x,y,z), R(x,y,z) >$ and the surface S is given by an equation $z = g(x,y)$, $(x,y)\in D$, then

$$\mathbf{n} = \frac{-\dfrac{\partial z}{\partial x}\mathbf{i} - \dfrac{\partial z}{\partial y}\mathbf{j} + \mathbf{k}}{\sqrt{\left[\dfrac{\partial z}{\partial x}\right]^2 + \left[\dfrac{\partial z}{\partial y}\right]^2 + 1}}$$

and

$$\iint\limits_S \mathbf{F}\cdot dS = \iint\limits_S \mathbf{F}\cdot\mathbf{n}\,dS = \iint\limits_D (P\mathbf{i} + Q\mathbf{j} + R\mathbf{k})\cdot\frac{-\dfrac{\partial z}{\partial x}\mathbf{i} - \dfrac{\partial z}{\partial y}\mathbf{j} + \mathbf{k}}{\sqrt{\left[\dfrac{\partial z}{\partial x}\right]^2 + \left[\dfrac{\partial z}{\partial y}\right]^2 + 1}}\sqrt{\left[\dfrac{\partial z}{\partial x}\right]^2 + \left[\dfrac{\partial z}{\partial y}\right]^2 + 1}\,dA$$

or

$$\boxed{\iint\limits_S \mathbf{F}\cdot dS = \iint\limits_D \left(-P\frac{\partial z}{\partial x} - Q\frac{\partial z}{\partial y} + R\right)dA}$$

If the surface S is given by vector function $\mathbf{r}(u,v) = x(u,v)\mathbf{i} + y(u,v)\mathbf{j} + z(u,v)\mathbf{k}$, $(u,v)\in D$, then

$$\mathbf{n} = \frac{\mathbf{r}_u\times\mathbf{r}_v}{|\mathbf{r}_u\times\mathbf{r}_v|}$$

and

$$\iint\limits_S \mathbf{F}\cdot dS = \iint\limits_S \mathbf{F}\cdot\mathbf{n}\,dS = \iint\limits_D \mathbf{F}\cdot\frac{\mathbf{r}_u\times\mathbf{r}_v}{|\mathbf{r}_u\times\mathbf{r}_v|}|\mathbf{r}_u\times\mathbf{r}_v|\,dA$$

or

$$\boxed{\iint\limits_S \mathbf{F}\cdot d\mathbf{S} = \iint\limits_D \mathbf{F}\cdot(\mathbf{r}_u\times\mathbf{r}_v)\,dA}$$

Example 2. Find the flux of the vector field $\mathbf{F} = x^2y\mathbf{i} - 3xy^2\mathbf{j} + 4y^3\mathbf{k}$ across the surface S, if S is the part of the elliptic paraboloid $z = x^2 + y^2 - 9$ that lies below the rectangle $0 \leq x \leq 2$, $0 \leq y \leq 1$ and has downward orientation.

Example 3. A fluid has density 1500 and velocity field

$$\mathbf{v} = -y\mathbf{i} + x\mathbf{j} + 2z\mathbf{k}$$

Find the rate of flow outward through the sphere $x^2 + y^2 + z^2 = 25$.