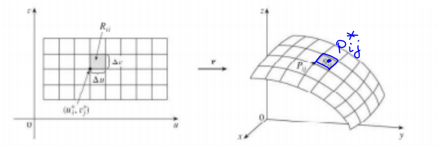
Section 16.7 **Surface integrals.**

Suppose f is a function of three variables whose domain include a surface S.



We divide S into patches S_{ij} with area ΔS_{ij}. We evaluate f at a point P_{ij}^* in each patch, multiply by the area ΔS_{ij}, and form the sum

$$\sum_{i=1}^{m}\sum_{j=1}^{n} f(P_{ij}^*)\Delta S_{ij}$$

We define the **surface integral of** f **over the surface** S as

$$\iint_S f(x,y,z)dS = \lim_{\|P\|\to 0}\sum_{i=1}^{m}\sum_{j=1}^{n} f(P_{ij}^*)\Delta S_{ij}$$

If the surface S is given by an equation $z = z(x,y)$, $(x,y)\in D$, then

$$\boxed{\iint_S f(x,y,z)dS = \iint_D f(x,y,z(x,y))\sqrt{\left[\frac{\partial z}{\partial x}\right]^2+\left[\frac{\partial z}{\partial y}\right]^2+1}\,dA}$$

If the surface S is given by vector function $\mathbf{r}(u,v) = x(u,v)\mathbf{i} + y(u,v)\mathbf{j} + z(u,v)\mathbf{k}$, $(u,v)\in D$, then

$$\boxed{\iint_S f(x,y,z)dS = \iint_D f(\mathbf{r}(u,v))|\mathbf{r}_u\times\mathbf{r}_v|\,dA}$$

where

$$\mathbf{r}_u = \frac{\partial x}{\partial u}\mathbf{i}+\frac{\partial y}{\partial u}\mathbf{j}+\frac{\partial z}{\partial u}\mathbf{k} \qquad \mathbf{r}_v = \frac{\partial x}{\partial v}\mathbf{i}+\frac{\partial y}{\partial v}\mathbf{j}+\frac{\partial z}{\partial v}\mathbf{k}$$

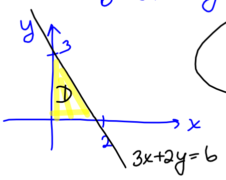
1. Evaluate $\iint_S y\,dS$, where S is the part of the plane $3x+2y+z=6$ that lies in the first octant.

$z = 6-3x-2y$, $z_x = -3$, $z_y = -2$

$$\iint_S y\,dS = \iint_D y\sqrt{1+[z_x]^2+[z_y]^2}\,dA$$

Parameter domain D:
$z=6-3x-2y$ and plug $z=0$:
$6 = 3x+2y$

$0 \le x \le \frac{6-2y}{3}$
$0 \le y \le 3$



$$= \int_0^3\int_0^{\frac{6-2y}{3}} y\sqrt{1+9+4}\,dx\,dy$$

$$= \sqrt{14}\int_0^3 yx\Big|_0^{\frac{6-2y}{3}} dy = \sqrt{14}\int_0^3 \frac{y(6-2y)}{3}\,dy$$

$$= \frac{2\sqrt{14}}{3}\int_0^3 y(3-y)\,dy = \frac{2\sqrt{14}}{3}\int_0^3 (3y-y^2)\,dy$$

$$= \frac{2\sqrt{14}}{3}\left[\frac{3y^2}{2}-\frac{y^3}{3}\right]_0^3 = \frac{2\sqrt{14}}{3}\,27\left(\frac{1}{2}-\frac{1}{3}\right) = \frac{\cancel{2}\sqrt{14}}{\cancel{3}}\cdot\frac{\cancel{27}}{\cancel{6}}$$

$$= \boxed{3\sqrt{14}}$$

2. Evaluate $\iint_S \sqrt{1+x^2+y^2}\, dS$, if S is given by vector equation $\mathbf{r}(u,v) = u\cos v\mathbf{i} + u\sin v\mathbf{j} + v\mathbf{k}$, $0 \le u \le 1$, $0 \le v \le \pi$.

$$\iint_S \sqrt{1+x^2+y^2}\, dS = \iint_D \sqrt{1+(u\cos v)^2+(u\sin v)^2}\; |\vec{r}_u \times \vec{r}_v|\, dA$$

$$\vec{r}(u,v) = \langle u\cos v,\ u\sin v,\ v\rangle$$

$$\vec{r}_u = \langle \cos v, \sin v, 0\rangle$$

$$\vec{r}_v = \langle -u\sin v,\ u\cos v, 1\rangle$$

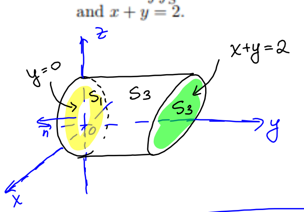
$$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \cos v & \sin v & 0 \\ -u\sin v & u\cos v & 1 \end{vmatrix} = \vec{i}(\sin v) - \vec{j}(\cos v) + \vec{k}(u\cos^2 v + u\sin^2 v)$$
$$= \vec{i}(\sin v) - \vec{j}(\cos v) + u\vec{k}$$

$$|\vec{r}_u \times \vec{r}_v| = \sqrt{\sin^2 v + \cos^2 v + u^2} = \sqrt{1+u^2}$$

$$\iint_S \sqrt{1+x^2+y^2}\, dA = \int_0^1 \int_0^\pi \sqrt{1+\underbrace{u^2\cos^2 v + u^2\sin^2 v}_{u^2(\cos^2 v+\sin^2 v)}}\ \sqrt{1+u^2}\, dv\, du = \int_0^1 \int_0^\pi \sqrt{1+u^2}\sqrt{1+u^2}\, dv\, du$$

$$= \int_0^1 \int_0^\pi (1+u^2)\, dv\, du = \int_0^1 (1+u^2)\, v\Big|_0^\pi du = \pi \int_0^1 (1+u^2)\, du = \pi\left(u+\frac{u^3}{3}\right)\Big|_0^1 = \boxed{\frac{4\pi}{3}}$$

3. Evaluate $\iint_S xy\, dS$, if S is a boundary of the region enclosed by the cylinder $x^2+z^2=1$ and the planes $y=0$ and $x+y=2$.

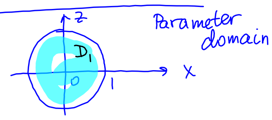


$$\iint_S = \iint_{S_1} + \iint_{S_2} + \iint_{S_3}$$

Plane $y=0$. Parametrization $\begin{cases} x=x \\ z=z \\ y=0 \end{cases}$

$$\iint_{S_1} xy\, dS = \iint_D x(0)|\vec{n}|\, dA = 0$$

Plane $x+y=2$. Parametrization $\begin{cases} x=x \\ y=2-x \\ z=z \end{cases}$



$$\iint_{S_2} xy\, dS = \iint_{D_1} x(2-x)|\vec{r}_x \times \vec{r}_z|\, dA$$

$$\vec{r}(x,z) = \langle x, 2-x, z\rangle$$

$$\vec{r}_x = \langle 1, -1, 0\rangle$$

$$\vec{r}_z = \langle 0, 0, 1\rangle$$

$$\vec{r}_x \times \vec{r}_z = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = -\vec{i} - \vec{j} = \langle -1, -1, 0\rangle$$

$$|\vec{r}_x \times \vec{r}_z| = \sqrt{1+1+0} = \sqrt{2}$$

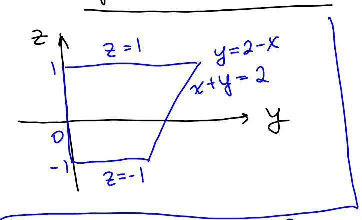
$$\iint_{S_2} xy\, dS = \sqrt{2}\iint_{D_1} (2x-x^2)\, dA = \left|\begin{array}{ll} x = r\cos\theta & 0 \le r \le 1 \\ z = r\sin\theta & 0 \le \theta \le 2\pi \\ dA = r\,dr\,d\theta & \end{array}\right|$$

$$= \sqrt{2}\int_0^{2\pi}\int_0^1 \left[2r\cos\theta - r^2\cos^2\theta\right] r\,dr\,d\theta = \sqrt{2}\int_0^{2\pi}\int_0^1 \left[2r^2\cos\theta - r^3\cos^2\theta\right] dr\,d\theta$$

$$= \sqrt{2}\int_0^{2\pi}\left[\frac{2r^3}{3}\cos\theta - \frac{r^4}{4}\cos^2\theta\right]_{r=0}^{r=1} d\theta = \frac{2\sqrt{2}}{3}\overbrace{\int_0^{2\pi}\cos\theta\, d\theta}^{0} - \frac{\sqrt{2}}{4}\int_0^{2\pi}\underbrace{\cos^2\theta}_{\frac{1+\cos 2\theta}{2}}\, d\theta$$

$$= -\frac{\sqrt{2}}{8}\int_0^{2\pi}(1+\cos 2\theta)\, d\theta = -\frac{\sqrt{2}}{8}\left(\theta + \frac{1}{2}\sin 2\theta\right)\Big|_0^{2\pi} = \boxed{-\frac{\pi\sqrt{2}}{4}}$$

Cylinder $x^2+z^2=1$



Parametrization: $\begin{cases} x=\cos\theta \\ y=y \\ z=\sin\theta \end{cases}$ $\quad 0\le\theta\le 2\pi$, $0\le y\le 2-x$, $0\le y\le 2-\cos\theta$

$\vec{r}(\theta,y)=\langle\cos\theta, y, \sin\theta\rangle$

$\vec{r}_\theta=\langle-\sin\theta, 0, \cos\theta\rangle$

$\vec{r}_y=\langle 0, 1, 0\rangle$

$$\vec{r}_\theta\times\vec{r}_y=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ -\sin\theta & 0 & \cos\theta\\ 0 & 1 & 0\end{vmatrix} = -\cos\theta\,\vec{i}-\sin\theta\,\vec{k} = \langle-\cos\theta, 0, -\sin\theta\rangle$$

$$|\vec{r}_\theta\times\vec{r}_y|=\sqrt{\cos^2\theta+\sin^2\theta}=1$$

$$\iint_{S_3} xy\,dS=\int_0^{2\pi}\int_0^{2-\cos\theta}(\cos\theta)y\,dy\,d\theta=\int_0^{2\pi}\cos\theta\,\frac{y^2}{2}\Big|_0^{2-\cos\theta}d\theta=\frac{1}{2}\int_0^{2\pi}\cos\theta\,(2-\cos\theta)^2\,d\theta$$

$$=\frac{1}{2}\int_0^{2\pi}\cos\theta\,(4-4\cos\theta+\cos^2\theta)\,d\theta=\frac{1}{2}\int_0^{2\pi}\Big(4\cos\theta-4\underbrace{\cos^2\theta}_{\frac{1+\cos 2\theta}{2}}+\underbrace{\cos^3\theta}_{\cos\theta\cdot\cos^2\theta}\Big)\,d\theta$$

$$=2\int_0^{2\pi}\cancel{\cos\theta}\,d\theta-2\int_0^{2\pi}(1+\cos 2\theta)\,d\theta+\frac{1}{2}\int_0^{2\pi}\cos^3\theta\,d\theta$$

$$=-2\left(\theta+\frac{1}{2}\sin 2\theta\right)\Big|_0^{2\pi}+\frac{1}{2}\int_0^{2\pi}\cancel{\cos\theta(1-\sin^2\theta)}\,d\theta \qquad \left|\begin{array}{l} u=\sin\theta \\ du=\cos\theta\,d\theta \\ 0\le u\le 0\end{array}\right|$$

$$=-4\pi$$

$$\iint_S xy\,dS=\iint_{S_1}+\iint_{S_2}+\iint_{S_3}=\boxed{-\frac{\sqrt{2}\pi}{4}-4\pi}$$

If a thin sheet has the shape of a surface S and the density at the point (x, y, z) is $\rho(x, y, z)$, then the **total mass** of the sheet is

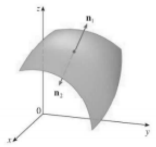
$$\boxed{m=\iint_S \rho(x, y, z)dS}$$

and the **center of mass** is $(\bar{x}, \bar{y}, \bar{z})$, where

$$\boxed{\bar{x}=\frac{1}{m}\iint_S x\rho(x, y, z)dS} \quad \boxed{\bar{y}=\frac{1}{m}\iint_S y\rho(x, y, z)dS} \quad \boxed{\bar{z}=\frac{1}{m}\iint_S z\rho(x, y, z)dS}$$

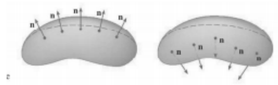
Oriented surfaces.

Let us consider a surface S that has a tangent plane at every point (x, y, z) on S (except at any boundary point).



$\vec{n}_2 = -\vec{n}_1$

There are two unit normal vectors $\mathbf{n}_1$ and $\mathbf{n}_2 = -\mathbf{n}_1$ at (x, y, z). If it is possible to chose a unit normal vector $\mathbf{n}$ at every such point (x, y, z) so that $\mathbf{n}$ varies continuously over S, then S is called an **oriented surface** and the given choice of $\mathbf{n}$ provides S with an **orientation**. There are two possible orientations for any orientable surface.



For a surface $z = z(x, y)$ the orientation is given by the unit normal vector

$$\boxed{\mathbf{n} = \frac{-\dfrac{\partial z}{\partial x}\mathbf{i} - \dfrac{\partial z}{\partial y}\mathbf{j} + \mathbf{k}}{\sqrt{\left[\dfrac{\partial z}{\partial x}\right]^2 + \left[\dfrac{\partial z}{\partial y}\right]^2 + 1}}}$$

$\vec{n} = \pm < z_x, z_y, -1 >$

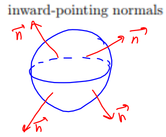
Since the $\mathbf{k}$-component is positive, this gives the *upward* orientation of the surface.

If S is a smooth orientable surface given in parametric form by a vector function $\mathbf{r}(u, v)$, then its orientation is given by a unit normal vector

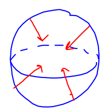
$$\boxed{\mathbf{n} = \frac{\mathbf{r}_u \times \mathbf{r}_v}{|\mathbf{r}_u \times \mathbf{r}_v|}}$$

$\vec{n} = \pm \vec{r}_u \times \vec{r}_v$

For a **closed surface**, the positive orientation is the one for which the normal vectors point *outward* from S, the inward-pointing normals give the negative orientation.



Positive orientation



Negative orientation

Surface integrals of vector fields.

Definition. If $\mathbf{F}$ is a continuous vector field defined on an oriented surface S with normal vector $\mathbf{n}$, then the **surface integral of $\vec{F}$ over** S is

$$\boxed{\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_S \mathbf{F} \cdot \mathbf{n}\, dS}$$

This integral is also called the **flux** of $\mathbf{F}$ across S.

If $\mathbf{F} =< P(x, y, z), Q(x, y, z), R(x, y, z) >$ and the surface S is given by an equation $z = g(x, y)$, $(x, y) \in D$, then

$$\mathbf{n} = \frac{-\dfrac{\partial z}{\partial x}\mathbf{i} - \dfrac{\partial z}{\partial y}\mathbf{j} + \mathbf{k}}{\sqrt{\left[\dfrac{\partial z}{\partial x}\right]^2 + \left[\dfrac{\partial z}{\partial y}\right]^2 + 1}}$$

and

$$\iint_S \mathbf{F} \cdot dS = \iint_S \mathbf{F} \cdot \mathbf{n}\, dS = \iint_D (P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}) \cdot \frac{-\dfrac{\partial z}{\partial x}\mathbf{i} - \dfrac{\partial z}{\partial y}\mathbf{j} + \mathbf{k}}{\sqrt{\left[\dfrac{\partial z}{\partial x}\right]^2 + \left[\dfrac{\partial z}{\partial y}\right]^2 + 1}} \sqrt{\left[\dfrac{\partial z}{\partial x}\right]^2 + \left[\dfrac{\partial z}{\partial y}\right]^2 + 1}\, dA$$

or

$$\boxed{\iint_S \mathbf{F} \cdot dS = \iint_D \left(-P\frac{\partial z}{\partial x} - Q\frac{\partial z}{\partial y} + R\right) dA}$$

If the surface S is given by vector function $\mathbf{r}(u, v) = x(u, v)\mathbf{i} + y(u, v)\mathbf{j} + z(u, v)\mathbf{k}$, $(u, v) \in D$, then

$$\mathbf{n} = \frac{\mathbf{r}_u \times \mathbf{r}_v}{|\mathbf{r}_u \times \mathbf{r}_v|}$$

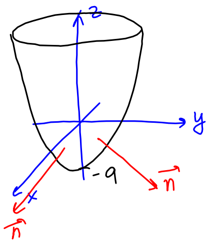
and

$$\iint_S \mathbf{F} \cdot dS = \iint_S \mathbf{F} \cdot \mathbf{n}\, dS = \iint_D \mathbf{F} \cdot \frac{\mathbf{r}_u \times \mathbf{r}_v}{|\mathbf{r}_u \times \mathbf{r}_v|} |\mathbf{r}_u \times \mathbf{r}_v|\, dA$$

or

$$\boxed{\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_D \mathbf{F} \cdot (\mathbf{r}_u \times \mathbf{r}_v)\, dA}$$

Example 2. Find the flux of the vector field $\mathbf{F} = x^2y\mathbf{i} - 3xy^2\mathbf{j} + 4y^3\mathbf{k}$ across the surface S, if S is the part of the elliptic paraboloid $z = x^2 + y^2 - 9$ that lies below the rectangle $\underbrace{0 \le x \le 2,\ 0 \le y \le 1}_{\text{parameter domain } D}$ and has downward orientation.



$$\text{flux} = \iint_S \vec{F}\cdot\vec{n}\,dS$$

$$\vec{F} = \langle x^2y,\ -3xy^2,\ 4y^3\rangle$$

$$S:\quad z = x^2+y^2-9$$

$$\vec{n} = \pm\langle z_x,\ z_y,\ -1\rangle$$

$$= + \langle 2x,\ 2y,\ -1\rangle$$

$$\vec{F}\cdot\vec{n} = \langle x^2y,\ -3xy^2,\ 4y^3\rangle\cdot\langle 2x, 2y, -1\rangle = 2x^3y - 6xy^3 - 4y^3$$

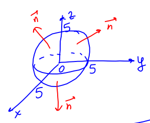
$$\text{flux} = \int_0^2\int_0^1 (2x^3y - 6xy^3 - 4y^3)\,dy\,dx = \int_0^2 \left[x^3y^2 - 6x\frac{y^4}{4} - \frac{4y^4}{4}\right]_0^1 dx$$

$$= \int_0^2 \left[x^3 - \frac{3}{2}x - 1\right]dx = \left[\frac{x^4}{4} - \frac{3}{2}\frac{x^2}{2} - x\right]_0^2 = 4 - 3 - 2 = \boxed{-1}$$

Example 3. A fluid has density 1500 and velocity field

$$\mathbf{v} = -y\mathbf{i} + x\mathbf{j} + 2z\mathbf{k} = \langle -y, x, 2z\rangle$$

Find the rate of flow outward through the sphere $x^2 + y^2 + z^2 = 25$.



$$\text{rate of flow} = \iint_S (1500)\,\vec{v}\cdot\vec{n}\,dS$$

Parametrization of the sphere $x^2+y^2+z^2=5$

$$0 \le \theta \le 2\pi \qquad 0 \le \varphi \le \pi$$

$$\begin{cases} x = 5\cos\theta\sin\varphi \\ y = 5\sin\theta\sin\varphi \\ z = 5\cos\varphi \end{cases}$$

$$\vec{r}(\theta,\varphi) = \langle 5\cos\theta\sin\varphi,\ 5\sin\theta\sin\varphi,\ 5\cos\varphi\rangle$$

$$\vec{r}_\theta = \langle -5\sin\theta\sin\varphi,\ 5\cos\theta\sin\varphi,\ 0\rangle$$

$$\vec{r}_\varphi = \langle 5\cos\theta\cos\varphi,\ 5\sin\theta\cos\varphi,\ -5\sin\varphi\rangle$$

$$\vec{r}_\theta\times\vec{r}_\varphi = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -5\sin\theta\sin\varphi & 5\cos\theta\sin\varphi & 0 \\ 5\cos\theta\cos\varphi & 5\sin\theta\cos\varphi & -5\sin\varphi \end{vmatrix} = \vec{i}(-25\cos\theta\sin^2\varphi) - \vec{j}(25\sin\theta\sin^2\varphi) + \vec{k}(\underbrace{-25\sin^2\theta\sin\varphi\cos\varphi - 25\cos^2\theta\sin\varphi\cos\varphi}_{-25\sin\varphi\cos\varphi})$$

$$= \langle -25\cos\theta\sin^2\varphi,\ -25\sin\theta\sin^2\varphi,\ -25\sin\varphi\cos\varphi\rangle$$

$0 \le \varphi \le \frac{\pi}{2}$ $\quad -25\sin\varphi\cos\varphi \le 0$

$$\vec{n} = - \langle -25\cos\theta\sin^2\varphi,\ -25\sin\theta\sin^2\varphi,\ -25\sin\varphi\cos\varphi\rangle$$

the third component must be positive for $0 < \varphi \le \frac{\pi}{2}$

$$\vec{n} = \langle 25\cos\theta\sin^2\varphi,\ 25\sin\theta\sin^2\varphi,\ 25\sin\varphi\cos\varphi\rangle$$

$$\vec{v} = \langle -y, x, 2z\rangle = \langle -5\sin\theta\sin\varphi,\ 5\cos\theta\sin\varphi,\ 10\cos\varphi\rangle$$

$$\vec{v}\cdot\vec{n} = \cancel{-125\cos\theta\sin\theta\sin^3\varphi} + \cancel{125\sin\theta\cos\theta\sin^3\varphi} + 250\sin\varphi\cos^2\varphi = 250\sin\varphi\cos^2\varphi$$

$$\iint_S 1500\,\vec{v}\cdot\vec{n}\,dS = \int_0^{2\pi}\int_0^{\pi} 1500\cdot 250\sin\varphi\cos^2\varphi\,d\varphi\,d\theta$$

$u = \cos\varphi$

$du = -\sin\varphi\,d\varphi$

$1 \le u \le -1$

$$= (1500)(250)\int_0^{2\pi}\int_{-1}^{1} u^2\,du\,d\theta = (1500)(250)\int_0^{2\pi} \frac{u^3}{3}\Big|_{-1}^{1} d\theta$$

$$= \boxed{(1500)(250)\frac{2}{3}(2\pi)}$$