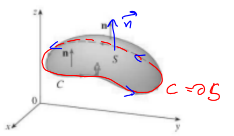
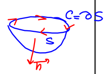
Section 16.8 **Stokes' Theorem.**

Stokes' Theorem. Let S be an oriented piecewise-smooth surface that is bounded by a simple, closed, piecewise-smooth boundary curve C with positive orientation. Let $\mathbf{F}$ be a vector field whose components have continuous partial derivatives on an open region in $\mathbb{R}^3$ that contains S. Then

$$\boxed{\oint_C \mathbf{F}\cdot d\mathbf{r} = \iint_S \operatorname{curl}\mathbf{F}\cdot d\mathbf{S}}$$

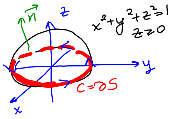




The Stokes' Theorem says that the line integral around the boundary curve of S of the tangential component of $\mathbf{F}$ is equal to the surface integral of the normal component of the curl of $\mathbf{F}$.

Example 1. Use Stokes' Theorem to evaluate $\iint_S \operatorname{curl}\mathbf{F}\cdot d\mathbf{S}$ if $\mathbf{F}(x,y,z) =< xyz, x, e^{xy}\cos(z) >$ and S is hemisphere $x^2+y^2+z^2=1$, oriented upward.

$$\iint_S \operatorname{curl}\vec F\cdot d\vec S = \oint_{C=\partial S}\vec F\cdot d\vec r = \int_a^b \vec F(\vec r(t))\cdot\vec r\,'(t)\,dt$$



$C: \begin{cases} x^2+y^2+z^2=1 \\ z=0\end{cases}$ or $\begin{cases} x^2+y^2=1 \\ z=0\end{cases}$

Parametrization of C: $\begin{cases} x=\cos\theta \\ y=\sin\theta \\ z=0\end{cases}$ $0\le\theta\le 2\pi$

$\vec r(\theta) = <\cos\theta, \sin\theta, 0>$

$\vec r\,'(\theta) = <-\sin\theta, \cos\theta, 0>$

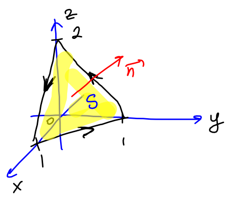
$\vec F = <xyz, x, e^{xy}\cos z>$

$\vec F(\vec r(\theta)) = <0, \cos\theta, e^{\cos\theta\sin\theta}\cos 0> = <0, \cos\theta, e^{\cos\theta\sin\theta}>$

$\vec F(\vec r(\theta))\cdot\vec r\,'(\theta) = <0, \cos\theta, e^{\cos\theta\sin\theta}>\cdot<-\sin\theta, \cos\theta, 0> = \cos^2\theta$

$$\iint_S \operatorname{curl}\vec F\cdot d\vec S = \int_0^{2\pi}\cos^2\theta\,d\theta = \int_0^{2\pi}\frac{1+\cos 2\theta}{2}\,d\theta = \left(\frac{\theta}{2}+\frac14\sin 2\theta\right)_0^{2\pi} = \boxed{\pi}$$

Example 2. Use Stokes' Theorem to evaluate $\oint_C \mathbf{F}\cdot d\mathbf{r}$ if $\mathbf{F}(x,y,z) =< z^2, y^2, xy >$ and C is the triangle with vertices (1,0,0), (0,1,0), and (0,0,2) and is oriented counterclockwise as viewed from above.



$$\oint_C \vec F\cdot d\vec r = \iint_S \operatorname{curl}\vec F\cdot d\vec S = \iint_D \operatorname{curl}\vec F\cdot\vec n\,dA$$

Plane S: $x+y+\frac{z}{2}=1$ or $z=2-2x-2y$

$\vec n = \pm <z_x, z_y, -1>$

$= - <-2,-2,-1>$ should be positive

$\vec n = <2,2,1>$

$$\operatorname{curl}\vec F = \begin{vmatrix} \vec i & \vec j & \vec k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z^2 & y^2 & xy \end{vmatrix} = \vec i\left(\underbrace{\frac{\partial}{\partial y}(xy)}_{x} - \underbrace{\frac{\partial}{\partial z}(y^2)}_{0}\right) - \vec j\left(\underbrace{\frac{\partial}{\partial x}(xy)}_{y} - \underbrace{\frac{\partial}{\partial z}(z^2)}_{2z}\right) + \vec k\left(\underbrace{\frac{\partial}{\partial x}(y^2)}_{0} - \underbrace{\frac{\partial}{\partial y}(z^2)}_{0}\right)$$

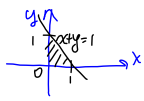
$\operatorname{curl}\vec F = <x, 2z-y, 0>$

$\operatorname{curl}\vec F\cdot\vec n = <x, 2z-y, 0>\cdot<2,2,1> = 2x+4z-2y$

$= 2x+4(2-2x-2y)-2y = 8-6x-10y$

$$\oint_C \vec F\cdot d\vec r = \iint_D (8-6x-10y)\,dA = \int_0^1\int_0^{1-x}(8-6x-10y)\,dy\,dx$$

Parameter domain:



$0\le x\le 1$

$0\le y\le 1-x$

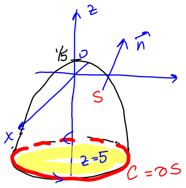
$$= \int_0^1\Big[(8-6x)y-5y^2\Big]_0^{1-x}dx = \int_0^1\left[(8-6x)(1-x)-5(1-x)^2\right]dx$$

$$= \int_0^1\left[8-8x-6x+6x^2-5(1-2x+x^2)\right]dx$$

$$= \int_0^1\left[3-12x+x^2\right]dx = \ldots$$

$z = \frac{1-x^2-y^2}{5}$

Example 3. Verify Stokes' Theorem for the surface S: $x^2+y^2+5z=1$, $z \geq -5$ (oriented by upward normal) and the vector field $\mathbf{F} = xz\mathbf{i} + yz\mathbf{j} + (x^2+y^2)\mathbf{k}$.



C: intersection of $z = \frac{1-x^2-y^2}{5}$ and $z=-5$

$-5 = \frac{1-x^2-y^2}{5} \Rightarrow -25 = 1-x^2-y^2$

$\begin{cases} x^2+y^2=26 \\ z=-5 \end{cases}$

Parametrization of C: $\begin{cases} x = \sqrt{26}\cos\theta \\ y = \sqrt{26}\sin\theta \\ z=-5 \end{cases}$ $\quad 0 \leq \theta \leq 2\pi$

$\oint_C \vec{F}\cdot d\vec{r} = \int_a^b \vec{F}(\vec{r}(t))\cdot\vec{r}\,'(t)\,dt$

$\vec{r}(\theta) = \langle \sqrt{26}\cos\theta, \sqrt{26}\sin\theta, -5\rangle$

$\vec{r}\,'(\theta) = \langle -\sqrt{26}\sin\theta, \sqrt{26}\cos\theta, 0\rangle$

$\vec{F} = \langle xz, yz, x^2+y^2\rangle$

$\vec{F}(\vec{r}(\theta)) = \langle -5\sqrt{26}\cos\theta, -5\sqrt{26}\sin\theta, 26\rangle$

$\vec{F}(\vec{r}(\theta))\cdot\vec{r}\,'(\theta) = +5(26)\cos\theta\sin\theta - 5(26)\sin\theta\cos\theta + 0$
$= \boxed{0}$

$\oint_C \vec{F}\cdot d\vec{r} = \boxed{0}$

$\iint_S \text{curl}\,\vec{F}\cdot d\vec{S} = \iint_D \text{curl}\,\vec{F}\cdot\vec{n}\,dA$

$S: \ z = \frac{1-x^2-y^2}{5}$

$\vec{n} = \pm\langle z_x, z_y, -1\rangle = \cancel{\pm}\, \ominus \langle -\frac{2x}{5}, -\frac{2y}{5}, -1\rangle$

$\vec{n} = \langle \frac{2x}{5}, \frac{2y}{5}, 1\rangle$

$\text{curl}\,\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xz & yz & x^2+y^2 \end{vmatrix} = \vec{i}\left(\frac{\partial}{\partial y}(x^2+y^2) - \frac{\partial}{\partial z}(yz)\right) - \vec{j}\left(\frac{\partial}{\partial x}(x^2+y^2) - \frac{\partial}{\partial z}(xz)\right)$
$+\vec{k}\left(\frac{\partial}{\partial x}(yz) - \frac{\partial}{\partial y}(xz)\right)$

$= \vec{i}(2y-y) - \vec{j}(2x-x) + \vec{k}\,0$

$= \langle y, -x, 0\rangle$

$\text{curl}\,\vec{F}\cdot\vec{n} = \langle y, -x, 0\rangle\cdot\langle \frac{2x}{5}, \frac{2y}{5}, 1\rangle$

$= \frac{2xy}{5} - \frac{2xy}{5} + 0 = 0$

$\iint_S \text{curl}\,\vec{F}\cdot\vec{n}\,dS = \boxed{0}$