Section 16.9 **Divergence Theorem.**

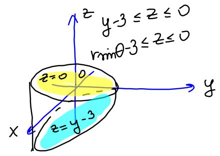
Divergence theorem. Let E be a simple solid region whose boundary surface S has positive (outward) orientation. Let $\mathbf{F}$ be a vector field whose component functions have continuous partial derivatives on an open region that contains E. Then

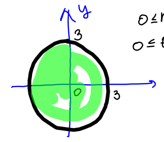
$$\boxed{\iint_S \mathbf{F}\cdot d\mathbf{S} = \iiint_E \operatorname{div} \mathbf{F}\, dV}$$

Example 1. Use the Divergence Theorem to calculate the surface integral $\iint_S \mathbf{F}\cdot d\mathbf{S}$ if $\mathbf{F} =< ye^{z^2}, y^2, e^{xy} >$ and S is the surface of the solid bounded by the cylinder $x^2+y^2=9$ and the planes $z=0$ and $z=y-3$.

$y-3 \le z \le 0$

$r\sin\theta - 3 \le z \le 0$





$0 \le r \le 3$

$0 \le \theta \le 2\pi$

$$\iint_S \vec{F}\cdot d\vec{S} = \iiint_E \operatorname{div}\vec{F}\, dV$$

$$\operatorname{div}\vec{F} = \frac{\partial}{\partial x}\left(ye^{z^2}\right) + \frac{\partial}{\partial y}\left(y^2\right) + \frac{\partial}{\partial z}\left(e^{xy}\right)$$

$$\operatorname{div}\vec{F} = 2y$$

$$\iint_S \vec{F}\cdot d\vec{S} = \iiint_E 2y\, dV$$

cylindrical coordinates

$z = z$

$x = r\cos\theta$

$y = r\sin\theta$

$dV = r\,dz\,dr\,d\theta$

$x^2+y^2=r^2$

$$= 2\int_0^{2\pi}\int_0^3\int_{r\sin\theta-3}^{0} \underbrace{r\sin\theta}_{y}\, \underbrace{r\,dz\,dr\,d\theta}_{dV} = 2\int_0^{2\pi}\int_0^3 r^2\sin\theta\; z\Big|_{r\sin\theta-3}^{0} dr\,d\theta$$

$$= 2\int_0^{2\pi}\int_0^3 r^2\sin\theta\,(3 - r\sin\theta)\,dr\,d\theta = 2\int_0^{2\pi}\int_0^3 \left(3r^2\sin\theta - r^3\sin^2\theta\right) dr\,d\theta$$

$$= 2\int_0^{2\pi}\left(\frac{3r^3}{3}\sin\theta - \frac{r^4}{4}\sin^2\theta\right)\Big|_0^3 d\theta = 2\int \cancel{27\sin\theta\, d\theta}^{\,0} - 2\cdot\frac{81}{4}\int^{2\pi} \underbrace{\sin^2\theta}_{\frac{1-\cos 2\theta}{2}}\, d\theta$$

$$= -2\cdot\frac{81}{8}\cdot\int_0^{2\pi}(1-\cos 2\theta)\,d\theta = -\frac{81}{4}\left(\theta - \cancel{\frac{1}{2}\sin 2\theta}^{\,0}\right)\Big|_0^{2\pi}$$

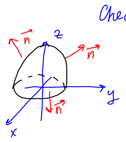
$$= \boxed{-\frac{81\pi}{2}}$$

Example 2. Verify the Divergence Theorem for the region

$$E = \{(x,y,z) : 0 \le z \le 9 - x^2 - y^2\}$$

and the vector field $\mathbf{F} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$

Check if $\iint_S \vec{F}\cdot d\vec{S} = \iiint_E \operatorname{div}\vec{F}\,dV$



$\boxed{\text{RHS}}$

$$\iiint_E \operatorname{div}\vec{F}\,dV$$

$$\vec{F} = \langle x, y, z\rangle$$

$$\operatorname{div}\vec{F} = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z) = 3$$

$$3\iiint_E dV =$$

cylindrical

$x = r\cos\theta$

$y = r\sin\theta$

$z = z$

$x^2+y^2=r^2$

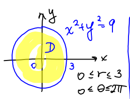
$dV = r\,dz\,dr\,d\theta$

$0 \le z \le 9 - x^2 - y^2$

$0 \le z \le 9 - r^2$

$0 \le r \le 3$

$0 \le \theta \le 2\pi$



$0 \le r \le 3$

$0 \le \theta \le 2\pi$

$$= 3\int_0^{2\pi}\int_0^3\int_0^{9-r^2} r\,dz\,dr\,d\theta$$

$$= 3\int_0^{2\pi}\int_0^3 rz\Big|_0^{9-r^2} dr\,d\theta$$

$$= 3\int_0^{2\pi}\int_0^3 r(9-r^2)\,dr\,d\theta$$

$$= 3\int_0^{2\pi}\int_0^3 (9r - r^3)\,dr\,d\theta$$

$$= 3(2\pi)\left(\frac{9r^2}{2} - \frac{r^4}{4}\right)\Big|_0^3 = 3\pi\left(81 - \frac{81}{2}\right) = \boxed{\frac{3\pi(81)}{2}}$$

$\boxed{\text{LHS}}$

$$\iint_S \vec{F}\cdot d\vec{S} = \iint_D \vec{F}\cdot\vec{n}\,dA$$

$z = 0$ $\quad \vec{n} = -\vec{k} = \langle 0, 0, -1\rangle$

Parametrization: $\begin{cases} x = x \\ y = y \\ z = 0 \end{cases}$

$$\vec{F} = \langle x, y, \cancel{z}^{\,0}\rangle = \langle x, y, 0\rangle$$

$$\vec{F}\cdot\vec{n} = \langle x, y, 0\rangle\cdot\langle 0, 0, -1\rangle = 0$$

$$\iint_{(z=0)} \vec{F}\cdot d\vec{S} = 0$$

$z = 9 - x^2 - y^2$

$$\vec{n} = \pm\langle z_x, z_y, -1\rangle = -\langle -2x, -2y, -1\rangle$$

$$\vec{n} = \langle 2x, 2y, 1\rangle$$

$$\vec{F} = \langle x, y, z\rangle = \langle x, y, 9 - x^2 - y^2\rangle$$

$$\vec{F}\cdot\vec{n} = \langle 2x, 2y, 1\rangle\cdot\langle x, y, 9 - x^2 - y^2\rangle$$

$$= 2x^2 + 2y^2 + 9 - x^2 - y^2 = x^2 + y^2 + 9$$

$$\iint_{(z=9-x^2-y^2)} \vec{F}\cdot d\vec{S} = \iint_D (x^2+y^2+9)\,dA$$

$x = r\cos\theta$

$y = r\sin\theta$

$dA = r\,dr\,d\theta$

$$= \int_0^{2\pi}\int_0^3 (r^2+9)\,r\,dr\,d\theta = \int_0^{2\pi}\int_0^3 (r^3 + 9r)\,dr\,d\theta$$

$$= \int_0^{2\pi}\left(\frac{r^4}{4} + \frac{9r^2}{2}\right)\Big|_0^3 d\theta = 2\pi\left(\frac{81}{4} + \frac{81}{2}\right) = \boxed{\frac{3\pi(81)}{2}}$$

$$\iint_S \vec{F}\cdot d\vec{S} = \boxed{\frac{3\pi(81)}{2}}$$

$\boxed{\text{RHS} = \text{LHS}.}$

1