Math 251. WEEK in REVIEW 1. Fall 2013

1. a) Find the center and radius of the sphere

$$x^2 + y^2 + z^2 + 4x + 6y - 10z + 2 = 0$$

$$(x^2+\overset{2\cdot 2}{4x}) + (y^2+\overset{2\cdot 3}{6y})+(z^2-\overset{5\cdot 2}{10z})+2=0$$

$$(x^2+4x+4)-4+(y^2+6y+9)-9+(z^2-10z+25)-25+2=0$$

$$(x+2)^2+(y+3)^2+(z-5)^2-36=0$$

$$(x+2)^2+(y+3)^2+(z-5)^2=36$$

$$\boxed{\begin{array}{l}\text{center } (-2,-3,5)\\ R=\sqrt{36}=6\end{array}}$$

b) Find an equation of the sphere given that it touches the $yz-$ plane and has the center at $(2, 1, 3)$.

Center $(2,1,3)$

R = distance from the center to the (yz)-plane

$R=2$

$$(x-2)^2+(y-1)^2+(z-3)^2=4$$

touches (xy)-plane

$$(x-2)^2+(y-1)^2+(z-3)^2=9$$

touches (xz)-plane

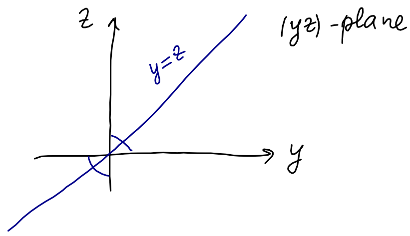
$$(x-2)^2+(y-1)^2+(z-\cancel{3})^2=1$$

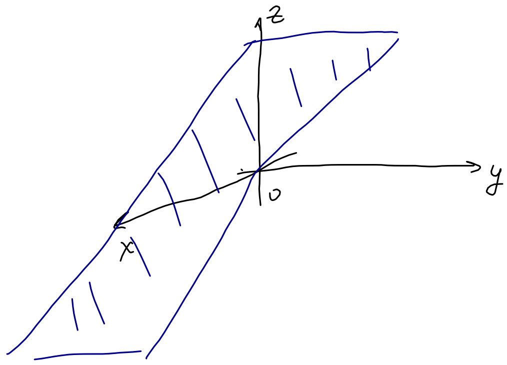
2. Describe in words the region of $\mathbb{R}^3$ represented by the equation or inequality.

(a) $y = z$

plane
passes through $(0,0,0)$
parallel to the x-axis

the plane contains the x-axis.





(b) $y > 2$ the set of all points that lie to the right of the plane $y=2$.

(c) $y^2 + z^2 \leq 4$

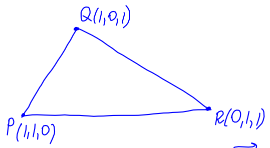
$y^2+z^2=4$ defines a circular cylinder of radius 2
the axis of the cylinder is the x-axis

$y^2+z^2 \leq 4$ is the set of points inside of the cylinder
and on the cylinder

(d) $x^2 + y^2 + z^2 - \overset{2\cdot 1}{2z} + 1 < 3$ $\quad x^2+y^2+(z^2-2z+1) < 3+1$

$x^2+y^2+(z-1)^2 < 4$ points inside the sphere with radius 2
and center $(0,0,1)$

3. Find the area of the triangle with the vertices $P(1,1,0)$, $Q(1,0,1)$, and $R(0,1,1)$.



$$A = \frac{1}{2}\left|\vec{PQ} \times \vec{PR}\right|$$

$$\vec{PQ} = \langle 0, -1, 1 \rangle$$

$$\vec{PR} = \langle -1, 0, 1 \rangle$$

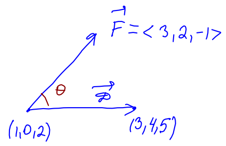
$$\vec{PQ} \times \vec{PR} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & -1 & 1 \\ -1 & 0 & 1 \end{vmatrix} = \vec{i}\begin{vmatrix} -1 & 1 \\ 0 & 1 \end{vmatrix} - \vec{j}\begin{vmatrix} 0 & 1 \\ -1 & 1 \end{vmatrix} + \vec{k}\begin{vmatrix} 0 & -1 \\ -1 & 0 \end{vmatrix}$$

$$= -\vec{i} - \vec{j} - \vec{k}$$

$$= \langle -1, -1, -1 \rangle$$

$$A = \frac{1}{2}\sqrt{1+1+1} = \boxed{\frac{\sqrt{3}}{2}}$$

4. a) A constant force of $\vec{F} = 3\vec{i} + 2\vec{j} - \vec{k}$ moves an object along the line segment from $(1,0,2)$ to $(3,4,5)$. Find the work done.



WORK = (FORCE)(DISTANCE)

DISTANCE VECTOR $\vec{D} = \langle 2, 4, 3 \rangle$

$$W = \vec{F} \cdot \vec{D} = \langle 3, 2, -1 \rangle \cdot \langle 2, 4, 3 \rangle$$

$$= 6 + 8 - 3 = 11$$

b) Find the angle between the force and the displacement vectors in question a).

$\vec{F} = <3, 2, -1>$

$\vec{D} = <2, 4, 3>$

$$\cos\theta = \frac{\vec{F}\cdot\vec{D}}{|\vec{F}|\cdot|\vec{D}|} = \frac{11}{\sqrt{9+4+1}\sqrt{4+16+9}} = \frac{11}{\sqrt{14}\sqrt{29}}$$

$$\theta = \cos^{-1}\left(\frac{11}{\sqrt{14}\sqrt{29}}\right)$$

5. Find two unit vectors orthogonal to both $\vec{a} = <1, 0, 1>$ and $\vec{b} = <2, 3, 4>$.

$$\vec{c} = \vec{a}\times\vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & 1 \\ 2 & 3 & 4 \end{vmatrix} = \vec{i}\begin{vmatrix} 0 & 1 \\ 3 & 4 \end{vmatrix} - \vec{j}\begin{vmatrix} 1 & 1 \\ 2 & 4 \end{vmatrix} + \vec{k}\begin{vmatrix} 1 & 0 \\ 2 & 3 \end{vmatrix}$$

$$= -3\vec{i} - 2\vec{j} + 3\vec{k}$$

$$= <-3, -2, 3>$$

$$\vec{u}_1 = \frac{\vec{c}}{|\vec{c}|} = \frac{<-3,-2,3>}{\sqrt{9+4+9}} = \boxed{\frac{1}{\sqrt{22}}<-3,-2,3>}$$

$$\vec{u}_2 = -\vec{u}_1 = \boxed{-\frac{1}{\sqrt{22}}<-3,-2,3>}$$

6. What restrictions must be made on b so that the vector $2\vec{i}+b\vec{j}$ is orthogonal to vector $-3\vec{i}+2\vec{j}+\vec{k}$? to vector $\vec{k}$?

$<2,b,0>$ want it to be orthogonal to $<-3,2,1>$

$<2,b,0>\cdot<-3,2,1> = -6+2b=0$

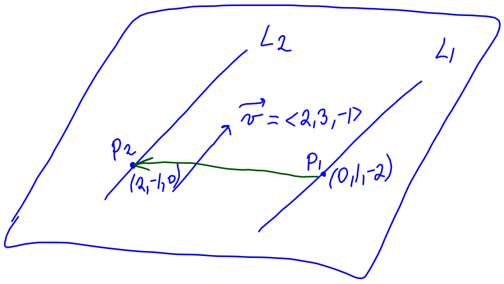
$\boxed{b=3}$

$<2,b,0>$ and $\vec{k}=<0,0,1>$ orthogonal for all b.

7. Find an equation for the plane containing two lines

$L_1: \vec{r} =< 0, 1, -2 > +t < 2, 3, -1 >$ and

$L_2: \vec{r} =< 2, -1, 0 > +t < 2, 3, -1 >$.



$L_1 \parallel L_2$

$\overrightarrow{P_1P_2} = <2,-2,2>$

$\vec{v} \nparallel \overrightarrow{P_1P_2}$

$\vec{n} = \vec{v} \times \overrightarrow{P_1P_2} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 3 & -1 \\ 2 & -2 & 2 \end{vmatrix}$

$= \vec{i}(6-2)-\vec{j}(4+2)+\vec{k}(-4-6)$

$= 4\vec{i}-6\vec{j}-10k$

Equation of the plane:

$4(x-0)-6(y-1)-10(z+2)=0$

$\boxed{4x-6y-10z-14=0}$

$4(x-2)-6(y+1)-10z=0$

$\boxed{4x-6y-10z-14=0}$

8. Find symmetric and parametric equations of the line through the point $M(2, 0, -3)$ and
   a) parallel to the x-axis;

line is parallel to $\vec{i} = \langle 1, 0, 0 \rangle$

Parametric equations:

$$x = 2 + t$$
$$y = 0 + 0t = 0$$
$$z = -3 + 0t = -3$$

$$\boxed{\begin{aligned} x &= 2+t \\ y &= 0 \\ z &= -3 \end{aligned}}$$

Symmetric equations: $\dfrac{x-2}{1} = \dfrac{y}{0} = \dfrac{z+3}{0}$

$$\boxed{\begin{aligned} &\frac{x-2}{1} = t \\ &y = 0 \\ &z = -3 \end{aligned}}$$

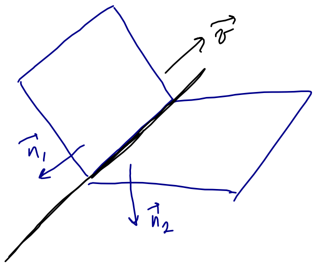
b) parallel to the line of intersection of the planes $3x - y + 2z - 7 = 0$ and $x + 3y - 2z - 3 = 0$.

Point $(2, 0, -3)$

$$\begin{cases} 3x - y + 2z - 7 = 0 \\ x + 3y - 2z - 3 = 0 \end{cases}$$

$\vec{n}_1 = \langle 3, -1, 2 \rangle$

$\vec{n}_2 = \langle 1, 3, -2 \rangle$



$$\vec{v} = \vec{n}_1 \times \vec{n}_2 = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & -1 & 2 \\ 1 & 3 & -2 \end{vmatrix}$$

$$= \vec{i}(2-6) - \vec{j}(-6-2) + \vec{k}(9+1)$$

$$= -4\vec{i} + 8\vec{j} + 10\vec{k}$$

$\langle 2, -4, 5 \rangle$

Parametric equations:

$$\boxed{\begin{aligned} x &= 2 - 4t \\ y &= 8t \\ z &= -3 + 10t \end{aligned}}$$

Symmetric equations: $\dfrac{x-2}{-4} = \dfrac{y}{8} = \dfrac{z+3}{10}$

9. a) Verify that the given planes

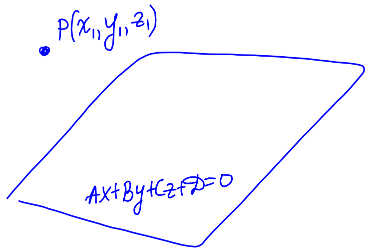
$2x + y - 3z + 4 = 0$ and $4x + 2y - 6z = 3$

are parallel and find the distance between them.

$2x + y - 3z + 4 = 0 \qquad \vec{n}_1 = \langle 2, 1, -3 \rangle$

$4x + 2y - 6z = 3 \qquad \vec{n}_2 = \langle 4, 2, -6 \rangle$

$\vec{n}_1 \parallel \vec{n}_2$, planes are parallel



$$\boxed{\text{Distance} = \frac{|Ax_1 + By_1 + Cz_1 + D|}{\sqrt{A^2 + B^2 + C^2}}}$$

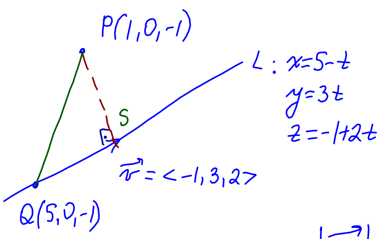
$2x + y - 3z + 4 = 0 \qquad x = 0,\ z = 0,\ y + 4 = 0,\ y = -4 \qquad P(0, -4, 0)$

Distance from $(0,-4,0)$ to the plane $4x + 2y - 6z - 3 = 0$

$$D = \frac{|4(0) + 2(-4) + (-6)(0) - 3|}{\sqrt{16 + 4 + 36}} = \boxed{\frac{11}{\sqrt{56}}}$$

b) Find the distance from the point $(1, 0, -1)$ to the line

$x = 5 - t$, $y = 3t$, $z = -1 + 2t$.



$\vec{PQ} = \langle 4, 0, 0 \rangle, \quad |\vec{PQ}| = 4$

$$|\vec{QS}| = |\text{comp}_{\vec{v}} \vec{QP}| = \left| \frac{\vec{v} \cdot \vec{QP}}{|\vec{v}|} \right| = \left| \frac{-4}{\sqrt{1+9+4}} \right| = \frac{4}{\sqrt{14}}$$

$\vec{QS} = \text{proj}_{\vec{v}} \vec{QP}$

$$|\vec{PS}| = \sqrt{16 - \frac{16}{14}} = \sqrt{\frac{208}{14}} = \boxed{\frac{\sqrt{104}}{7}}$$

$$A_\triangle = \frac{1}{2} \cdot h \cdot a = \frac{1}{2} |\vec{PS}|\,|\vec{QS}|$$

$$|\vec{PS}| = \frac{2 A_\triangle}{|\vec{QS}|} = \frac{2 \cdot \frac{1}{2} |\vec{QP} \times \vec{QS}|}{|\vec{QS}|} = \frac{|\vec{QP} \times \vec{QS}|}{|\vec{QS}|} = \frac{\sqrt{104}}{7}$$

c) Verify that the given lines

$\vec{n}_1 = <3,-1,2>$ $\quad$ $\vec{n}_2 = <1,3,-2>$

$\{3x - y + 2z - 7 = 0, x + 3y - 2z - 3 = 0\}$ and

$\frac{x+7}{2} = \frac{y-5}{-4} = \frac{z-9}{-5}$ $\quad \parallel <2,-4,-5>$

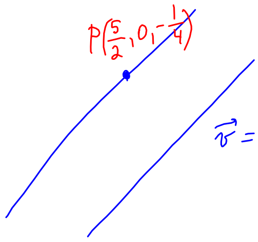
are parallel and find the distance between them.

line 1 is parallel to $\quad \vec{n}_1 \times \vec{n}_2 = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & -1 & 2 \\ 1 & 3 & -2 \end{vmatrix} = -4\vec{i} + 8\vec{j} + 10\vec{k}$

$= <-4, 8, 10>$

$<2,-4,-5> \parallel <-4,8,10>$

lines are parallel



$\vec{v} = <2,-4,-5>$

$\begin{cases} 3x - y + 2z - 7 = 0 \\ x + 3y - 2z - 3 = 0 \end{cases}$

$y = 0$

$+\begin{cases} 3x + 2z - 7 = 0 \\ x - 2z - 3 = 0 \end{cases}$

$4x - 10 = 0 \longrightarrow x = \frac{5}{2}$

$2z = x - 3 = \frac{5}{2} - 3 = -\frac{1}{2}$

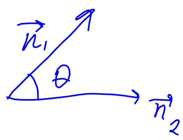
$z = -\frac{1}{4}$

Find the distance from $(\frac{5}{2}, 0, -\frac{1}{4})$ to the line $\frac{x+7}{2} = \frac{y-5}{-4} = \frac{z-9}{-5}$ as we did in part b).

10. Find an angle between the planes

$2x + 2y - z = 4$ and $6x - 3y + 2z = 5$.

$\vec{n}_1 = <2,2,-1>$ $\qquad$ $\vec{n}_2 = <6,-3,2>$



$\cos\theta = \frac{\vec{n}_1 \cdot \vec{n}_2}{|\vec{n}_1|\,|\vec{n}_2|} = \frac{12 - 6 - 2}{\sqrt{4+4+1}\sqrt{36+9+4}} = \frac{4}{21}$

$\theta = \cos^{-1}\left(\frac{4}{21}\right)$

11. Determine whether the line through $(2,1,0)$ and $(1,0,1)$ and the line $\frac{x}{2} = \frac{y-3}{-1} = \frac{z+5}{2}$ are parallel, skew, or intersecting. If they are intersecting, find the point of intersection and the angle between the lines. If they are parallel or skew, find the distance between the lines.

$\vec{v}_1 = \langle -1,-1,1 \rangle$, $P_1(2,1,0)$

$L_1: \begin{cases} x = 2-t \\ y = 1-t \\ z = t \end{cases}$

$\vec{v}_2 = \langle 2,-1,2 \rangle$ $\quad P_2(0,3,-5)$

$L_2: \frac{x}{2} = \frac{y-3}{-1} = \frac{z+5}{2} = s$

Parametric equations for L_2: $\begin{cases} x = 2s \\ y = 3-s \\ z = -5+2s \end{cases}$

$\begin{matrix} x: \\ y: \\ z: \end{matrix} \begin{cases} 2-t = 2s \\ 1-t = 3-s \\ t = -5+2s \end{cases}$ $\qquad - \begin{cases} 2-t = 2s \\ 1-t = 3-s \end{cases}$

$1 = 2s-(3-s)$

$1 = 3s-3$

$3s = 4$, $s = \frac{4}{3}$

$t = 2-2s = 2-\frac{8}{3} = -\frac{2}{3}$

$-\frac{2}{3} \overset{?}{=} -5+2\left(\frac{4}{3}\right) = -5+\frac{8}{3} = -\frac{7}{3}$

$-\frac{2}{3} \neq -\frac{7}{3}$

$\boxed{\text{SKEW}}$

Write the equation of a plane that contains L_1:

$\vec{n} = \vec{v}_1 \times \vec{v}_2 = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -1 & -1 & 1 \\ 2 & -1 & 2 \end{vmatrix} = -\vec{i}+4\vec{j}+3\vec{k}$

$-1(x-2)+4(y-1)+3z = 0$

$-x+4y+3z-2 = 0$

Distance from $(0,3,-5)$ to $-x+4y+3z-2=0$

$D = \frac{|12-15-2|}{\sqrt{1+16+9}} = \boxed{\frac{5}{\sqrt{26}}}$

12. Find the point at which the line $x=2-t$, $y=1+3t$, $z=4t$ intersects the plane $2x-y+z=2$

$2(2-t) - (1+3t) + 4t = 2$

solve for t: $\quad 4-2t-1-3t+4t = 2$

$-t = 2+1-4$

$-t = -1$

$t = 1$

$x = 2-1 = 1$

$y = 1+3(1) = 4$

$z = 4(1) = 4$

Point of intersection $\boxed{(1,4,4)}$