Math 251. WEEK in REVIEW 10. Fall 2013

1. Evaluate the surface integral $\iint_S (x^2 z + y^2 z)\,dS$ where $\quad z(x^2+y^2)$

(a) S is the part of the plane $z = 4 + x + y$ that lies inside the cylinder $x^2 + y^2 = 4$.

if S is given by $z = g(x,y)$, then

$$\iint_S f(x,y,z)\,dS = \iint_D f(x,y,z(x,y))\sqrt{[z_x]^2+[z_y]^2+1}\,dA$$

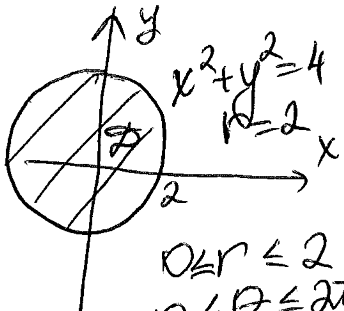
where D is the parameter domain.

$$z = 4+x+y$$

$$f(x,y,z) = z(x^2+y^2)$$

$$f(x,y,z(x,y)) = (4+x+y)(x^2+y^2)$$

D: $x^2+y^2 \le 4$



$$z_x = z_y = 1$$

$$dS = \sqrt{[z_x]^2+[z_y]^2+1}\,dA = \sqrt{3}\,dA$$

$0 \le r \le 2$

$0 \le \theta \le 2\pi$

$$\iint_S z(x^2+y^2)\,dS = \iint_D (4+x+y)(x^2+y^2)\sqrt{3}\,dA$$

Polar coord.

$x = r\cos\theta$

$y = r\sin\theta$

$x^2+y^2 = r^2$

$dA = r\,dr\,d\theta$

$$= \int_0^{2\pi}\int_0^2 (4 + r\cos\theta + r\sin\theta)\,r^2\sqrt{3}\,r\,dr\,d\theta$$

$$= \sqrt{3}\int_0^{2\pi}\int_0^2 \left(4r^3 + r^4(\cos\theta+\sin\theta)\right)dr\,d\theta$$

$$= \sqrt{3}\int_0^{2\pi}\left(r^4 + \frac{r^5}{5}(\cos\theta+\sin\theta)\right)\Big|_{r=0}^{r=2} d\theta$$

$$= \sqrt{3}\int_0^{2\pi}\left(16 + \frac{32}{5}(\cos\theta+\sin\theta)\right)d\theta$$

$$= \sqrt{3}\left[16\theta + \frac{32}{5}(\sin\theta - \cos\theta)\right]_0^{2\pi} = \boxed{32\sqrt{3}\,\pi}$$

$\frac{\pi}{2} \le \varphi \le \pi$

(b) S is the portion of the sphere $x^2 + y^2 + z^2 = 25$ such that $z \le 0$, $y \ge 0$. $0 \le \theta \le \pi$

Parametrize the sphere.

spherical coord.

$$\begin{cases} x = \rho \cos\theta \sin\varphi \\ y = \rho \sin\theta \sin\varphi \\ z = \rho\cos\varphi \end{cases} \qquad x^2+y^2+z^2=\rho^2$$

on the sphere $x^2+y^2+z^2 = 25 = \rho^2$

$\rho = 5$.

Param. equations of the sphere: $\begin{cases} x = 5\cos\theta\sin\varphi \\ y = 5\sin\theta\sin\varphi \\ z = 5\cos\varphi \end{cases}$

$$x^2+y^2 = 25\sin^2\varphi$$

Vector equation of the sphere:

$$\vec{r}(\theta,\varphi) = \langle 5\cos\theta\sin\varphi, 5\sin\theta\sin\varphi, 5\cos\varphi\rangle.$$

$$\iint_S f(x,y,z)\,dS = \iint_D f(\vec{r}(\theta,\varphi))\,|\vec{r}_\theta \times \vec{r}_\varphi|\,dA$$

$$\vec{r}_\theta = \langle -5\sin\theta\sin\varphi, 5\cos\theta\sin\varphi, 0\rangle$$

$$\vec{r}_\varphi = \langle 5\cos\theta\cos\varphi, 5\sin\theta\cos\varphi, -5\sin\varphi\rangle$$

$$\vec{r}_\varphi \times \vec{r}_\theta = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -5\sin\theta\sin\varphi & 5\cos\theta\sin\varphi & 0 \\ 5\cos\theta\cos\varphi & 5\sin\theta\cos\varphi & -5\sin\varphi \end{vmatrix}$$

$$= \vec{i}\, 25\sin^2\varphi\cos\theta + 25\sin\theta\sin^2\varphi\,\vec{j} + \vec{k}\,(25\cos^2\theta\cos\varphi\sin\varphi + 25\sin^2\theta\cos\varphi\sin\varphi)$$

$$= \langle 25\sin^2\varphi\cos\theta, 25\sin\theta\sin^2\varphi, 25\cos\varphi\sin\varphi\rangle$$

$$|\vec{r}_\varphi \times \vec{r}_\theta| = 25\sqrt{\sin^4\varphi\cos^2\theta + \sin^4\varphi\sin^2\theta + \cos^2\varphi\sin^2\varphi}$$

$$= 25\sqrt{\sin^4\varphi(\cos^2\theta+\sin^2\theta) + \cos^2\varphi\sin^2\varphi}$$

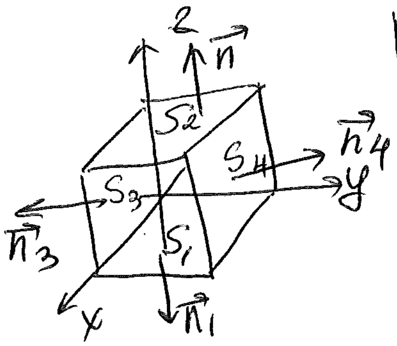
$$= 25\sqrt{\sin^4\varphi + \cos^2\varphi\sin^2\varphi} = 25\sin\varphi$$

$$\iint_S z(x^2+y^2)\,dS = 25(25)(5)\iint_D \sin^3\varphi\cos\varphi\,dA = 3125\int_0^\pi\int_{\pi/2}^{\pi}\sin^3\varphi\cos\varphi\,d\varphi\,d\theta$$

$$= 3125\pi\int_{\pi/2}^{\pi}\sin^3\varphi\cos\varphi\,d\varphi = 3125\pi\int_1^0 u^3\,du = \boxed{-\frac{3125\pi}{4}}$$

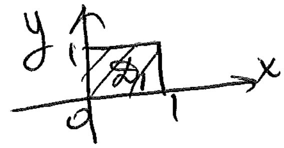
$u = \sin\varphi$

2. Evaluate the surface integral $\iint_S (x+y+z)dS$ where S is the surface of the cube defined by the inequalities $0 \le x, y, z \le 1$. Closed surface, normal is outward. Param. equations for the sides:



S_1: $z=0$ $(0\le x\le 1, 0\le y\le 1) = \mathcal{D}_1$

$\vec{n}_1 = \langle 0,0,-1\rangle$

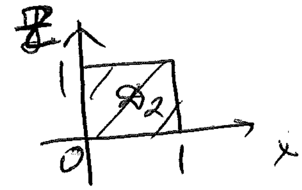


S_2: $z=1$, $(0\le x\le 1, 0\le y\le 1) = \mathcal{D}_1$

$\vec{n}_2 = \langle 0,0,1\rangle$

S_3: $y=0$, $(0\le z\le 1, 0\le x\le 1) = \mathcal{D}_2$

$\vec{n}_3 = \langle 0,-1,0\rangle$

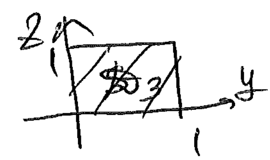


S_4: $y=1$, $(0\le z\le 1, 0\le x\le 1) = \mathcal{D}_2$

S_5: $x=0$ (back side).

$(0\le y\le 1, 0\le z\le 1) = \mathcal{D}_3$

$\vec{n}_5 = \langle -1,0,0\rangle$



S_6: $x=1$ (front side.)

$(0\le y\le 1, 0\le z\le 1) = \mathcal{D}_3$

$\vec{n}_6 = \langle 1,0,0\rangle$.

$|\vec{n}_1| = |\vec{n}_2| = \ldots = |\vec{n}_6| = 1$, $dS = |\vec{n}|dA = dA$.

$$\iint_S (x+y+z)dS = \iint_{S_1} + \iint_{S_2} + \ldots + \iint_{S_6}$$

$$= \iint_{\mathcal{D}_1} (x+y+0)dA + \iint_{\mathcal{D}_1} (x+y+1)dA + \iint_{\mathcal{D}_2} (x+0+z)dA + \iint_{\mathcal{D}_2} (x+1+z)dA$$

$$+ \iint_{\mathcal{D}_3} (0+y+z)\,dA + \iint_{\mathcal{D}_3} (1+y+z)\,dA$$

$$= \iint_{\mathcal{D}_1} (2x+2y+1)dA + \iint_{\mathcal{D}_2} (2x+2z+1)\,dA + \iint_{\mathcal{D}_3} (2y+2z+1)dA$$

$$= \int_0^1\int_0^1 (2x+2y+1)\,dx\,dy + \int_0^1\int_0^1 (2x+2z+1)\,dx\,dz + \int_0^1\int_0^1 (2y+2z+1)\,dy\,dz$$

$$= \int_0^1 (x^2+2yx+x)\Big|_{x=0}^{x=1} dy + \int_0^1 (x^2+2xz+z)\Big|_{x=0}^{x=1} dz + \int_0^1 (y^2+2yz+y)\Big|_{y=0}^{y=1} dz$$

$$= \int_0^1 (2+2y)\,dy + \int_0^1 (2+2z)\,dz + \int_0^1 (2+2z)\,dz = \left[2y+y^2\right]_0^1 + \left[4z+2z^2\right]_0^1 = \boxed{9}$$

3. Find the mass of the lamina that is the portion of the surface $z = 10 - x^2/2$ between the planes $x = 0$, $x = 1$, $y = 0$, $y = 1$, if the density is $\rho(x, y, z) = x$.

$$m = \iint_S \rho(x,y,z)\,dS = \iint_S x\,dS$$

$\mathcal{D}$: $0 \le x \le 1$, $0 \le y \le 1$

$$z = 10 - \frac{x^2}{2}, \quad z_x = -x, \quad z_y = 0$$

$$dS = \sqrt{[z_x]^2 + [z_y]^2 + 1}\,\cancel{dA}\ \sqrt{x^2+1}\,dA$$

$$m = \iint_{\mathcal{D}} x\sqrt{x^2+1}\,dA = \int_0^1\int_0^1 x\sqrt{x^2+1}\,dx\,dy$$

$$= \frac{1}{2}\int_0^1 2x\sqrt{x^2+1}\,dx$$

$u = x^2+1$
$du = 2x\,dx$
$x=0 \to u=1$
$x=1 \to u=2$

$$= \frac{1}{2}\int_1^2 \sqrt{u}\,du$$

$$= \frac{1}{2}\cdot\frac{2}{3}\,u^{3/2}\Big]_1^2$$

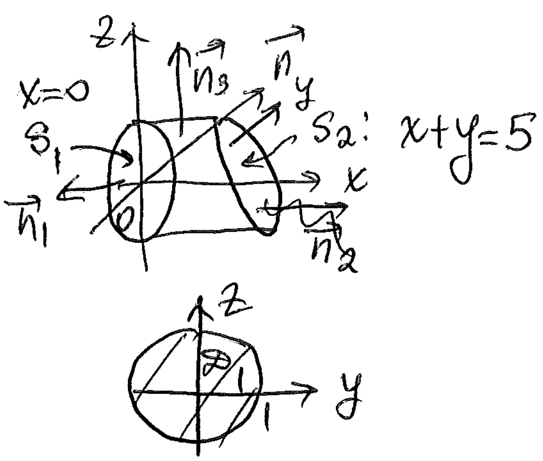
$$= \boxed{\frac{1}{3}(2\sqrt{2}-1)}$$

4. Find flux of the vector field $\vec{F} = < x, y, 1 >$ across the surface S which is the boundary of the region enclosed by the cylinder $y^2 + z^2 = 1$ and the planes $x = 0$ and $x + y = 5$. Use positive (outward) orientation for S.

$$\text{Flux} = \iint_S \vec{F} \cdot d\vec{S} = \iint_{\mathcal{D}} \vec{F} \cdot \vec{n}\, dA$$

$\vec{n}$ is the normal vector of S.

$S = S_1 \cup S_2 \cup S_3$



S_1: $x = 0$, $\cancel{z}^2 + y^2 \le 1 = \mathcal{D}_1$

$\vec{n}_1 = < -1, 0, 0 >$

$< x, y, 1 > \cdot < -1, 0, 0 > = -x$

since on S_1 $x = 0$, then

$< x, y, 1 > \cdot < -1, 0, 0 > = -x = 0.$

$$\iint_{S_1} \vec{F} \cdot d\vec{S} = 0$$

S_2: $x + y = 5$ $\quad (y^2 + z^2 \le 1) = \mathcal{D}_1$

Param. equation of the surface.

$$\begin{cases} x = 5 - y \\ y = y \\ z = z \end{cases}$$

Vector equation of the surface:

$$\vec{r}(y, z) = < 5 - y, y, z >$$

The normal vecton $\vec{n} = \oplus \vec{r}_y \times \vec{r}_z$

$\vec{r}_y = < -1, 1, 0 >$, $\vec{r}_z = < 0, 0, 1 >$

$$\vec{r}_y \times \vec{r}_z = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = \vec{i} + \vec{j} + 0 = < 1, 1, 0 >$$

$\vec{n} = < 1, 1, 0 >$

$$< x, y, 1 > \cdot < 1, 1, 0 > = \cancel{x} + y = \overbrace{5 - y}^{x} + y = 5$$

$$\iint_{S_2} \vec{F} \cdot \vec{n}\, dS = \iint_{\mathcal{D}_1} 5\, dA = 5(\text{area of } \mathcal{D}_1) = \boxed{5\pi}$$

S_3: $y^2+z^2=1$

the lateral surface of the cylinder.

$0 \le x \le 5-y$ $\to D_3$.

$\left(\begin{array}{l} 0 \le x \le 5-\cos\theta \\ 0\le\theta\le 2\pi \end{array}\right)$

Parametrize the cylinder.

cylindrical coord.

$$\begin{cases} x = x \\ y = r\cos\theta \\ z = r\sin\theta \end{cases} \qquad y^2+z^2=r^2$$

since on the cylinder $y^2+z^2=1$, then $r=1$

Param. equations of the cylinder:

$$\begin{array}{l} x=x \\ y=\cos\theta \\ z=\sin\theta \end{array}, \qquad \left(\begin{array}{l} 0\le x\le 5-\cos\theta \\ 0\le\theta\le 2\pi \end{array}\right) = D_3$$

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vector equation of the cylinder:

$\vec{r}(x,\theta) = \langle x, \cos\theta, \sin\theta \rangle$

$\vec{r}_x = \langle 1,0,0 \rangle$, $\vec{r}_\theta = \langle 0, -\sin\theta, \cos\theta \rangle$

$\vec{n} = \pm\, \vec{r}_x \times \vec{r}_\theta$

$$\vec{r}_x \times \vec{r}_\theta = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & 0 \\ 0 & -\sin\theta & \cos\theta \end{vmatrix}$$

$= 0 - \vec{j}\cos\theta + \vec{k}\sin\theta$

$= \langle 0, -\cos\theta, -\sin\theta \rangle$

$\vec{n} = \langle 0, \cos\theta, \sin\theta \rangle$

$\langle x, y, 1\rangle \cdot \langle 0, \cos\theta, \sin\theta\rangle = \cancel{\langle x, \cos\theta, 1\rangle \cdot \langle 0, \ldots}$

$= \langle x, \cos\theta, 1\rangle \cdot \langle 0, \cos\theta, \sin\theta\rangle$

$= \cos^2\theta + \sin\theta$

$$\iint_{S_3} \vec{F}\cdot d\vec{S} = \iint_{D_3} (\cos^2\theta + \sin\theta)\, dA = \int_0^{2\pi}\int_0^{5-\cos\theta} (\cos^2\theta + \sin\theta)\, dx\, d\theta$$

$$= \int_0^{2\pi} (5-\cos\theta)(\cos^2\theta + \sin\theta)\, d\theta$$

$$= \int_0^{2\pi} (5\cos^2\theta - \cos^3\theta + 5\sin\theta - \cos\theta\sin\theta)\, d\theta$$

$$= 5\int_0^{2\pi} \cos^2\theta \, d\theta + 5\cancelto{0}{\int_0^{2\pi} \sin\theta \, d\theta} - \int_0^{2\pi} \cos^3\theta \, d\theta - \cancelto{0}{\int_0^{2\pi} \cos\theta \sin\theta \, d\theta}$$

$$= \frac{5}{2}\int_0^{2\pi} (1+\cos 2\theta)\, d\theta - \int_0^{2\pi} \cos\theta \cdot \cos^2\theta \, d\theta - \frac{1}{2}\cancelto{0}{\int_0^{2\pi} \sin 2\theta \, d\theta}$$

$$= \frac{5}{2}\left[\theta + \frac{1}{2}\sin 2\theta\right]_0^{2\pi} - \cancelto{0}{\int_0^{2\pi} \cos\theta\,(1-\sin^2\theta)\, d\theta}$$

$$\left| \begin{array}{l} u = \sin\theta \\ du = \cos\theta \, d\theta \\ \theta = 0 \rightarrow u = 0 \\ \theta = 2\pi \rightarrow u = 0 \end{array} \right|$$

$$= \frac{5}{2}(2\pi)$$

$$= \boxed{5\pi}$$

$$\text{Flux} = \iint_{S_1} + \iint_{S_2} + \iint_{S_3} = 5\pi + 5\pi = \boxed{10\pi}$$

5. Evaluate the surface integral $\iint_S < x, y, 1 > \cdot d\vec{S}$ where S is the portion of the paraboloid $z = 1 - x^2 - y^2$ in the first octant, oriented by downward normals.

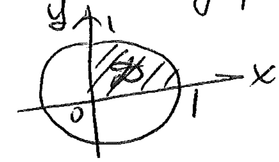
$z = 1 - x^2 - y^2$

$\vec{n} = \pm < z_x, z_y, -1 >$

$z_x = -2x, \quad z_y = -2y$

$\vec{n} = < -2x, -2y, -1 >.$

Projection of the paraboloid onto xy-plane



$$\iint_S <x, y, 1> \cdot d\vec{S} = \iint_D (-2x^2 - 2y^2 - 1)\, dA$$

Polar coord.

$x = r\cos\theta$

$y = r\sin\theta$

$dA = r\,dr\,d\theta$

$x^2 + y^2 = r^2$

$0 \le r \le 1$

$0 \le \theta \le \pi/2.$

$$= \int_0^{\pi/2}\int_0^1 (-2r^2 - 1)\, r\,dr\,d\theta$$

$$= -\int_0^{\pi/2}\int_0^1 (2r^3 + r)\,dr\,d\theta$$

$$= -\frac{\pi}{2}\left(\frac{2r^4}{4} + \frac{r^2}{2}\right)\Big|_0^1$$

$$= \boxed{-\frac{\pi}{2}}$$