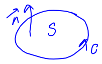
Math 251. WEEK in REVIEW 11. Fall 2013

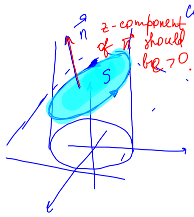
1. Use Stokes' Theorem to evaluate $\oint_C \vec{F}\cdot d\vec{r}$ where $\vec{F}(x,y,z) = < 3z, 5x, -2y >$ and C is the ellipse in which the plane $z = y+3$ intersects the cylinder $x^2+y^2=4$, with positive orientation as viewed from above.

Stokes' Theorem.

S is an oriented piece-wise smooth surface that is bounded by a simple, closed, piecewise smooth curve C with positive orientation. $\vec{F}$ is a vector field. Then



$$\oint_C \vec{F}\cdot d\vec{r} = \iint_S \operatorname{curl}\vec{F}\cdot d\vec{S}$$

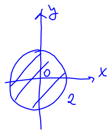


$$\operatorname{curl}\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 3z & 5x & -2y \end{vmatrix} = \vec{i}(-2) - \vec{j}(-3) + \vec{k}(5)$$
$$= <-2, 3, 5>$$

Parametrization for S:

$$\begin{cases} x = x \\ y = y \\ z = y+3 \end{cases}, \quad z_x = 0, \ z_y = 1$$

Parameter domain: $x^2+y^2 \le 4 : \mathcal{D}$



$\vec{n} = \pm < z_x, z_y, -1 >$

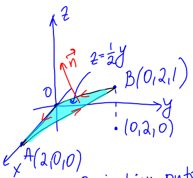
$= < 0, -1, 1 >$

$$\iint_S \operatorname{curl}\vec{F}\cdot d\vec{S} = \iint_{\mathcal{D}} \overbrace{\operatorname{curl}\vec{F}\cdot\vec{n}}^{2}\, dA$$

$\operatorname{curl}\vec{F}\cdot\vec{n} = <-2,3,5>\cdot<0,-1,1> = -3+5 = 2$

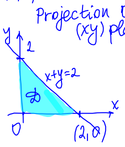
$$2\iint_{\mathcal{D}} dA = 2(\text{area of } \mathcal{D}) = 2(4\pi) = \boxed{8\pi}$$

2. Find the work performed by the forced field $\vec{F} = \langle -3y^2, 4z, 6x \rangle$ on a particle that traverses the triangle C in the plane $z = \frac{1}{2}y$ with vertices $A(2,0,0)$, $B(0,2,1)$, and $O(0,0,0)$ with a counterclockwise orientation looking down the positive z-axis.



Projection onto (xy) plane

$0 \le x \le 2-y$
$0 \le y \le 2$



$$W = \oint_C \vec{F}\cdot d\vec{r} = \iint_S \operatorname{curl}\vec{F}\cdot d\vec{S} = \iint_{\mathcal{D}} \operatorname{curl}\vec{F}\cdot\vec{n}\, dA$$

$\vec{n} = \pm < z_x, z_y, -1 >$, $z = \frac{1}{2}y$

$= \pm < 0, \frac{1}{2}, -1 >$

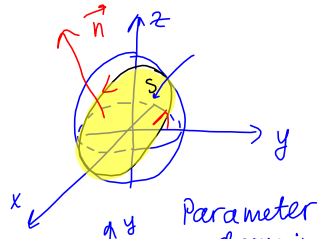
$= < 0, -\frac{1}{2}, 1 >$

$$\operatorname{curl}\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ -3y^2 & 4z & 6x \end{vmatrix} = \vec{i}(-4) - \vec{j}(6) + \vec{k}(+6y)$$
$$= <-4, -6, +6y>$$

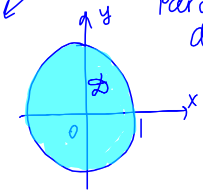
$\operatorname{curl}\vec{F}\cdot\vec{n} = 3+6y$

$$W = \int_0^2 \int_0^{2-y} (3+6y)\,dx\,dy$$
$$= \int_0^2 (3+6y)(2-y)\,dy$$
$$= \int_0^2 (6 - 3y + 12y - 6y^2)\,dy$$
$$= \int_0^2 (6 + 9y - 6y^2)\,dy$$
$$= \left[6y + \frac{9y^2}{2} - \frac{6y^3}{3} \right]_0^2$$
$$= 12 + 18 - 16$$
$$= \boxed{14}$$

3. Evaluate $I = \oint_C \vec{F} \cdot d\vec{r}$ if $\vec{F} = \langle 2y + 3e^x, z - y^8, x + \ln(z^2+1) \rangle$ and C is the curve of intersection of the plane $x + y + z = 0$ and sphere $x^2 + y^2 + z^2 = 1$. (Orient C to be counterclockwise when viewed from above).



Parameter domain:



Portion of the plane $z = -x - y$ inside the sphere $x^2+y^2+z^2=1$.

$\vec{n} = \pm \langle z_x, z_y, -1 \rangle$

$= \pm \langle -1, -1, -1 \rangle$

$= \langle 1, 1, 1 \rangle$

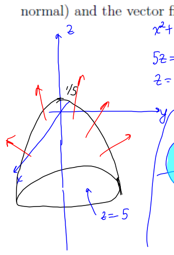
$$\text{curl } \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 2y+3e^x & z-y^8 & x+\ln(z^2+1) \end{vmatrix} = -\vec{i} - \vec{j} - 2\vec{k}$$

$= \langle -1, -1, -2 \rangle$

$\text{curl } \vec{F} \cdot \vec{n} = -4$

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_S \text{curl } \vec{F} \cdot d\vec{S} = \iint_D (-4)\, dA = -4(\text{area of } D) = \boxed{-4\pi}$$

4. Verify Stokes' Theorem for the surface S: $x^2 + y^2 + 5z = 1$, $z \geq -5$ (oriented by upward normal) and the vector field $\vec{F} = xz\vec{i} + yz\vec{j} + (x^2+y^2)\vec{k}$.

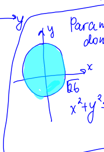


$x^2+y^2+5z=1$

$5z = 1 - x^2 - y^2$

$z = \frac{1}{5}(1 - x^2 - y^2)$

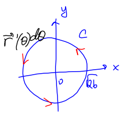
Parameter domain



$x^2+y^2 \leq 26$

$\oint_C F \cdot d\vec{r} \overset{?}{=} \iint_S \text{curl } \vec{F} \cdot d\vec{S}$

$\boxed{LHS}$ $\oint_C F \cdot d\vec{r} = \int_0^{2\pi} F(\vec{r}(\theta)) \cdot \vec{r}\,'(\theta)\, d\theta$



curve C:

$\begin{cases} x^2+y^2+5z=1 \\ z=-5 \end{cases}$

$x^2+y^2+5(-5)=1$

$x^2+y^2=26$

Parametrization for C.

$x = \sqrt{26}\cos\theta$

$y = \sqrt{26}\sin\theta$, $0 \leq \theta \leq 2\pi$

$z = -5$

$\vec{r}(\theta) = \langle \sqrt{26}\cos\theta, \sqrt{26}\sin\theta, -5 \rangle$

$\vec{r}\,'(\theta) = \langle -\sqrt{26}\sin\theta, \sqrt{26}\cos\theta, 0 \rangle$

$F(x,y,z) = \langle xz, yz, x^2+y^2 \rangle$

$\vec{F}(\vec{r}(\theta)) = \langle -5\sqrt{26}\cos\theta, -5\sqrt{26}\sin\theta, 26 \rangle$

$\oint_C \vec{F} \cdot d\vec{r} = \int_0^{2\pi} \langle -5\sqrt{26}\cos\theta, -5\sqrt{26}\sin\theta, 26 \rangle \cdot \langle -\sqrt{26}\sin\theta, \sqrt{26}\cos\theta, 0 \rangle\, d\theta$

$= \int_0^{2\pi} (5(26)\cos\theta\sin\theta - 5(26)\sin\theta\cos\theta)\, d\theta$

$= \boxed{0}$

$\boxed{RHS}$ $\iint_S \text{curl } \vec{F} \cdot d\vec{S} = \iint_D \underbrace{\text{curl } \vec{F} \cdot \vec{n}}_{0}\, dA = \boxed{0}$

$S: z = \frac{1}{5}(1 - x^2 - y^2)$

$\vec{n} = \pm \langle z_x, z_y, -1 \rangle$

$= \langle \frac{2}{5}x, \frac{2}{5}y, 1 \rangle$

$$\text{curl } \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ xz & yz & x^2+y^2 \end{vmatrix}$$

$= \vec{i}(2y - y) - \vec{j}(2x - x) + \vec{k}(0)$

$= \langle y, -x, 0 \rangle$

$\text{curl } \vec{F} \cdot \vec{n} = \langle \frac{2}{5}x, \frac{2}{5}y, 1 \rangle \cdot \langle y, -x, 0 \rangle$

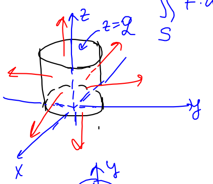
$= \frac{2}{5}xy - \frac{2}{5}xy + 0 = 0$

$\boxed{LHS = RHS}$

5. Use the Divergence Theorem to compute $\iint_S \vec{F}\cdot d\vec{S}$, where $\vec{F} = x^3\vec{\imath} + y^3\vec{\jmath} + z^3\vec{k}$ and S is the surface of the region enclosed by $x^2+y^2=1$ and the planes $z=0$, $z=2$.

The Divergence Theorem. E is a simple solid region whose boundary surface S has positive (outward) orientation. $\vec{F}$ is a vector field. Then

$$\iint_S \vec{F}\cdot d\vec{S} = \iiint_E \operatorname{div}\vec{F}\, dV$$

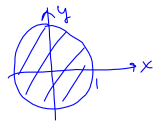


$$\operatorname{div}\vec{F} = \underbrace{3x^2+3y^2}_{3r^2} + 3z^2 = 3(r^2+z^2)$$

cylindrical coord.

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \\ dV = r\,dz\,dr\,d\theta \end{cases}$$

$0 \le z \le 2$
$0 \le r \le 1$
$0 \le \theta \le 2\pi$



$$= 3\int_0^{2\pi}\int_0^1\int_0^2 (r^2+z^2)\, r\,dz\,dr\,d\theta$$

$$= 3(2\pi)\int_0^1 \left[r^3 z + r\frac{z^3}{3}\right]_{z=0}^{z=2} dr$$

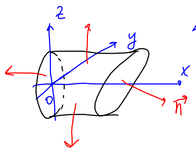
$$= 6\pi\int_0^1 \left[2r^3 + \frac{8}{3}r\right] dr$$

$$= 6\pi\left(\frac{2r^4}{4} + \frac{8r^2}{6}\right)\Big|_0^1$$

$$= 6\pi\underbrace{\left(\frac{1}{2}+\frac{4}{3}\right)}_{\frac{3+8}{6}}$$

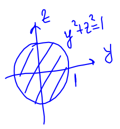
$$= \boxed{11\pi}$$

6. Use the Divergence Theorem to find flux of the vector field $\vec{F} = \langle x, y, 1\rangle$ across the surface S which is the boundary of the region enclosed by the cylinder $y^2+z^2=1$ and the planes $x=0$ and $x+y=5$.



$$\text{flux} = \iint_S \vec{F}\cdot d\vec{S} = \iiint_E \operatorname{div}\vec{F}\, dV = 2\iiint_E dV$$

$\vec{F} = \langle x, y, 1\rangle$
$\operatorname{div}\vec{F} = 2$



cylindrical coord.

$$\begin{cases} x = x \\ y = r\cos\theta \\ z = r\sin\theta \\ dV = r\,dx\,dr\,d\theta \end{cases}$$

$0 \le x \le 5-y$
$5 - r\cos\theta$
$0 \le r \le 1$
$0 \le \theta \le 2\pi$

$$\text{flux} = 2\int_0^{2\pi}\int_0^1\int_0^{5-r\cos\theta} r\,dx\,dr\,d\theta$$

$$= 2\int_0^{2\pi}\int_0^1 \overbrace{r(5-r\cos\theta)}^{5r - r^2\cos\theta}\, dr\,d\theta$$

$$= 2\int_0^{2\pi}\left(5\frac{r^2}{2} - \frac{r^3}{3}\cos\theta\right)\Big|_{r=0}^{r=1} d\theta$$

$$= 2\int_0^{2\pi}\left(\frac{5}{2} - \frac{1}{3}\cos\theta\right)d\theta$$

$$= 2\left[\frac{5}{2}\theta - \frac{1}{3}\sin\theta\right]_{\theta=0}^{\theta=2\pi}$$

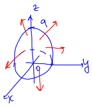
$$= \boxed{10\pi}$$

7. Verify the Divergence Theorem for the region

$$E = \{(x, y, z) : 0 \le z \le 9 - x^2 - y^2\}$$

and the vector field $\vec{F} = \vec{r} = x\vec{\imath} + y\vec{\jmath} + z\vec{k}$
$= \langle x, y, z \rangle$

$$\iint_S \vec{F} \cdot d\vec{S} \overset{?}{=} \iiint_E \operatorname{div} \vec{F}\, dV$$



$\boxed{LHP}$

$$\iint_S = \iint_{S_1} + \iint_{S_2}$$

S_1: $z = 9 - x^2 - y^2$

$$\iint_{S_1} \vec{F} \cdot d\vec{S} = \iint_D \vec{F} \cdot \vec{n}\, dA$$

$\vec{n} = \pm \langle z_x, z_y, -1 \rangle$

$= \langle 2x, 2y, 1 \rangle$

$\vec{F} = \langle x, y, \overset{= 9-x^2-y^2}{z} \rangle$

$= \langle x, y, 9 - x^2 - y^2 \rangle$

$\vec{F} \cdot \vec{n} = 2x^2 + 2y^2 + 9 - x^2 - y^2$

$= x^2 + y^2 + 9$

$= \iint_D (x^2 + y^2 + 9)\, dA$

Polar coordinates:
$x = r\cos\theta$ $\quad 0 \le r \le 3$
$y = r\sin\theta$ $\quad 0 \le \theta \le 2\pi$
$dA = r\,dr\,d\theta$

$$= \int_0^{2\pi} \int_0^3 (r^2 + 9)\, r\, dr\, d\theta$$

$$= 2\pi \left(\frac{r^4}{4} + \frac{9r^2}{2} \right)_0^3$$

$$= 2\pi \left(\frac{81}{4} + \frac{81}{2} \right)$$

$$= 2\pi \cdot \frac{3 \cdot 81}{4}$$

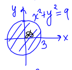
$$= \boxed{\frac{243\pi}{2}}$$

S_2: $z = 0$
$\vec{n} = \langle 0, 0, -1 \rangle$
$\vec{F} = \langle x, y, \overset{=0}{z} \rangle$
$= \langle x, y, 0 \rangle$
$\vec{F} \cdot \vec{n} = 0$
$\iint_{S_2} \vec{F} \cdot d\vec{S} = 0$

$\boxed{RHP}$ $\quad \operatorname{div} \vec{F} = 3$

cylindrical coord.
$x = r\cos\theta$
$y = r\sin\theta$
$z = z$
$dV = r\,dz\,dr\,d\theta$

$9 - r^2$
$0 \le z \le 9 - x^2 - y^2$
$0 \le r \le 3$
$0 \le \theta \le 2\pi$



$$= 3\int_0^{2\pi}\int_0^3\int_0^{9-r^2} r\, dz\, dr\, d\theta$$

$$= 6\pi \int_0^3 r(9 - r^2)\, dr$$

$$= 6\pi \left(9\frac{r^2}{2} - \frac{r^4}{4} \right)_0^3$$

$$= 6\pi \left(\frac{81}{2} - \frac{81}{4} \right)$$

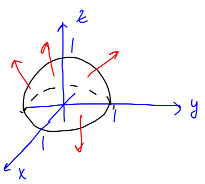
$$= 6\pi \cdot \frac{81}{4} = \boxed{\frac{243\pi}{2}}$$

$\boxed{LHS = RHS}$

8. Apply the Divergence Theorem to compute $\iint_S \vec{F} \cdot d\vec{S}$ for the vector field

$$\vec{F}(x, y, z) = \langle x^3 + \sin(yz), y^3, y + z^3 \rangle$$

over the complete boundary S of the solid hemisphere $\{(x, y, z) : x^2 + y^2 + z^2 \le 1, z \ge 0\}$ with outward normal.



$$\iint_S \vec{F} \cdot d\vec{S} = \iiint_E \operatorname{div} \vec{F}\, dV$$

$\operatorname{div} \vec{F} = 3x^2 + 3y^2 + 3z^2 = 3\rho^2$

spherical coordinates:
$x = \rho\cos\theta\sin\varphi$
$y = \rho\sin\theta\sin\varphi$
$z = \rho\cos\varphi$
$dV = \rho^2\sin\varphi\, d\rho\, d\varphi\, d\theta$

$0 \le \rho \le 1$
$0 \le \varphi \le \pi/2$
$0 \le \theta \le 2\pi$

$$= \int_0^{2\pi} \int_0^{\pi/2} \int_0^1 3\rho^2 \rho^2 \sin\varphi\, d\rho\, d\varphi\, d\theta$$

$$= 2\pi \int_0^{\pi/2} \sin\varphi\, d\varphi \int_0^1 3\rho^4\, d\rho$$

$$= 2\pi \left(-\cos\varphi\right)_0^{\pi/2} \left. \frac{3\rho^5}{5} \right|_0^1$$

$$= \boxed{\frac{6\pi}{5}}$$