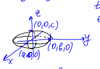
Math 251. WEEK in REVIEW 2. Fall 2013

1) $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ ellipsoid

traces in planes $x=k, y=k, z=k$ are ellipses

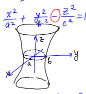


2) $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$ hyperboloid of one sheet

traces in $z=k$ are ellipses

traces in $y=k$ and $x=k$ are hyperbolas

negative term indicates the axis of the hyperboloid



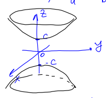
3) $-\frac{x^2}{a^2}-\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ hyperboloid of two sheets.

positive term indicates the axis of the hyperboloid

Is not defined for $|z|<c$.

traces in $z=k$ are ellipses

traces in $y=k$ and $z=k$ are hyperbolas

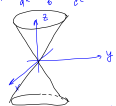


4) $\frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{z^2}{c^2}$ cone

Traces in $z=k$ are ellipses

Traces in $x=0$ and $y=0$ are pairs of lines

Traces in $x=k$ and $y=k$ $(k\neq 0)$ are hyperbolas

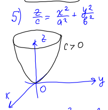


5) $\frac{z}{c}=\frac{x^2}{a^2}+\frac{y^2}{b^2}$ elliptic paraboloid

variable raised to the 1st power indicates the axis of the paraboloid

traces in horizontal planes are ellipses

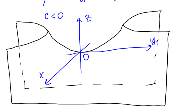
traces in vertical planes are parabolas



6) $\frac{z}{c}=\frac{x^2}{a^2}-\frac{y^2}{b^2}$ - hyperbolic paraboloid

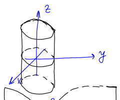
traces in horizontal planes are hyperbolas

traces in vertical planes are parabolas

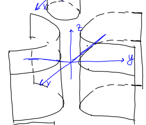


7) Quadric cylinders.

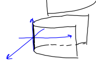
a) $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ elliptic cylinder



b) $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ hyperbolic cylinder



c) $y=ax^2$ parabolic cylinder



$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ ellipse; $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ hyperbola; $y=ax^2$ or $x=by^2$ parabola

if $a=b$, then it is a circle

1. Find the traces of the given surface in the planes $x=k$, $y=k$, $z=k$.

(a) $y-x^2-9z^2=0$

$x=k$: $y-k^2-9z^2=0$ or $y=9z^2+k^2$. parabola

$y=k$: $k-x^2-9z^2=0$ or $x^2+9z^2=k$ ellipse

$z=k$: $y-x^2-9k^2=0$ or $y=x^2+9k^2$ parabola

$16k^2-9\geq 0$ $\quad |k|\geq\frac{3}{4}$

The surface is not defined for $|x|<\frac{3}{4}$

(b) $16x^2-y^2-z^2=9$

$x=k$: $16k^2-y^2-z^2=9$ OR $y^2+z^2=16k^2-9$ circle

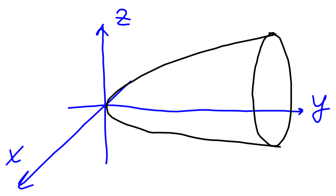
$y=k$: $16x^2-k^2-z^2=9$ or $16x^2-z^2=9+k^2$ hyperbola

$z=k$: $16x^2-y^2-k^2=9$ or $16x^2-y^2=9+k^2$ hyperbola

2. Sketch each of the following:

(a) $y - x^2 - 9z^2 = 0$

$y = x^2 + 9z^2$ elliptic paraboloid



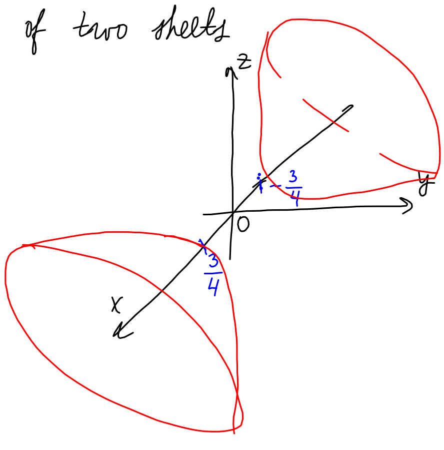
(b) $\underline{16x^2 - y^2 - z^2 = 9}$ hyperboloid of two sheets

9

$$\frac{16x^2}{9} - \frac{y^2}{9} - \frac{z^2}{9} = 1$$

$$\frac{x^2}{\left(\frac{9}{16}\right)} - \frac{y^2}{9} - \frac{z^2}{9} = 1$$

$$\sqrt{\frac{9}{16}} = \pm\frac{3}{4}$$



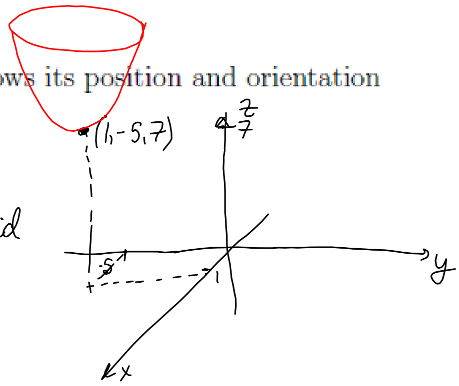
3. Identify the surface and make a rough sketch that shows its position and orientation

(a) $z = (x-1)^2 + (y+5)^2 + 7$

$z-7 = (x-1)^2 + (y+5)^2$ elliptic paraboloid

$(1,-5,7)$ vertex of the paraboloid

the axis of the paraboloid is parallel to the z-axis



(b) $4x^2 - y^2 + (z-4)^2 = 20$ $(0,0,4)$

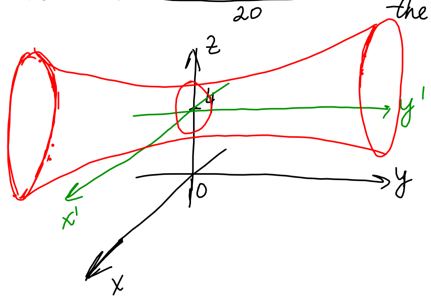
hyperboloid of one sheet

the axis of the paraboloid is parallel to the y-axis.

$$\frac{4x^2}{20} - \frac{y^2}{20} + \frac{(z+4)^2}{20} = 1$$

z-intersepts are $\pm\sqrt{20}$

x-intersepts are $\pm\frac{\sqrt{20}}{2}$



(c) $x^2 + y^2 + z + 6x - 2y + 10 = 0$

Complete squares:

$(x^2+6x) + (y^2-2y) + z + 10 = 0$

$(x^2+6x+9) - 9 + (y-2y+1) - 1 + z + 10 = 0$

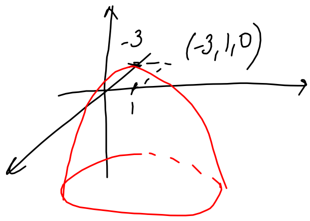
$(x+3)^2 + (y-1)^2 + z = 0$

OR

$(x+3)^2 + (y-1)^2 = -z$ elliptic paraboloid

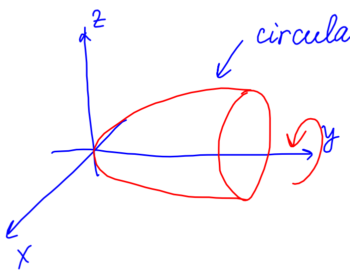
vertex $(-3,1,0)$

faces downward



4. Find an equation of the surface generated by revolving the curve given by $y = 25x^2$ and $z = 0$ about the y axis.

$\begin{cases} y = 25x^2 \\ z = 0 \end{cases}$ — the trace of the surface in the (xy)-plane



$y = a(x^2+z^2)$, a is an unknown constant

Find the trace of $y = a(x^2+z^2)$ in the plane $z=0$:

$y = ax^2 = 25x^2$

$\boxed{a = 25}$

$\boxed{y = 25(x^2+z^2)}$

5. Find the domain of $\vec{r}(t) = < \ln(4-t^2), \sqrt{1+t}, \sin(\pi t) >$.

$f(t) = \ln(4-t^2)$

$4-t^2>0$

or $t^2-4<0$

$-2<t<2$

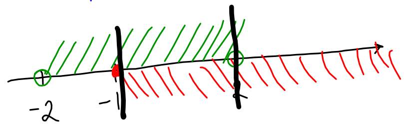
$g(t) = \sqrt{1+t}$

$1+t \geq 0$

$t \geq -1$

$h(t) = \sin \pi t$

defined for all t.



Domain is $\boxed{[-1, 2)}$

6. Find a vector equation for the curve of intersection of the surfaces $x = y^2$ and $z = x$ in terms of the parameter $y = t$.

an equation of the curve: $\begin{cases} x = y^2 \\ z = x \end{cases}$ if $\begin{cases} x = y^2 = t^2 \\ y = t \\ z = x = t^2 \end{cases}$ parametric equations of the surface

$\boxed{\vec{r}(t) = < t^2, t, t^2 >}$ vector equation of the surface

7. Does the graph of the vector-function $\vec{r}(t) = \left\langle \frac{1-t^2}{t}, \frac{t+1}{t}, t \right\rangle$ lie in the plane $x - y + z = -1$?

$x(t) = \frac{1-t^2}{t}$, $y(t) = \frac{t+1}{t}$, $z(t) = t$ parametric equations of the curve

$\frac{1-t^2}{t} - \frac{t+1}{t} + t \overset{?}{=} -1$

$\frac{\cancel{1} - \cancel{t^2} - t - \cancel{1} + \cancel{t^2}}{t}$

$-\frac{t}{t}$

$-1 = -1$

$\boxed{\text{YES}}$

8. Find the points where the curve $\vec{r}(t) =< 1-t, t^2, t^2 >$ intersects the plane $5x - y + 2z = -1$.

$x = 1-t$
$y = t^2$
$z = t^2$ parametric equations of the curve

$5(1-t) - t^2 + 2t^2 = -1$ solve for t.
$5 - 5t + t^2 = -1$
$t^2 - 5t + 6 = 1$

$t_1 = 2$ $\qquad$ $t_2 = 3$

$\vec{r}(2) = \boxed{(-1, 4, 4)}$ $\qquad$ $\vec{r}(3) = \boxed{(-2, 9, 9)}$

points of intersection

9. Find parametric equations of the line tangent to the graph of $\vec{r}(t) =< e^{-t}, t^3, \ln t >$ at the point $t = 1$.

a vector equation of the tangent line is $\boxed{<x,y,z> = \vec{r}(1) + t\vec{r}\,'(1)}$

$\vec{r}(1) = < e^{-1}, 1, \cancel{\ln 1}^{0} >$

$\vec{r}\,'(t) = < -e^{-t}, 3t^2, \frac{1}{t} >$

$\vec{r}\,'(1) = < -e^{-1}, 3, 1 >$

$< x, y, z > = < e^{-1}, 1, 0 > + t < -e^{-1}, 3, 1 >$

Parametric equations: $\boxed{\begin{array}{l} x = e^{-1} - te^{-1} \\ y = 1 + 3t \\ z = t \end{array}}$

10. Find symmetric equations of the line tangent to the graph of $\vec{r}(t) = \left\langle t^2, 4 - t^2, -\frac{3}{1+t} \right\rangle$ at the point $(4, 0, 3)$.

Find t such that $\vec{r}(t) = < 4, 0, 3 >$

$t^2 = 4$
$4 - t^2 = 0$ $\quad > \quad t = \pm 2$
$-\frac{3}{1+t} = 3$ $\qquad \boxed{t = -2}$

$\vec{r}(-2) = < 4, 0, 3 >$

$\vec{r}\,'(t) = < 2t, -2t, \frac{3}{(1+t)^2} >$

$\vec{r}\,'(-2) = < -4, 4, 3 >$

symmetric equations are: $\boxed{\frac{x-4}{-4} = \frac{y}{4} = \frac{z-3}{3}}$

11. Let

$$\vec{r}_1(t) =< \arctan t, t, -t^4 >$$

and

$$\vec{r}_2(t) = \left\langle t^2 - t, 2\ln t, \frac{\sin(2\pi t)}{2\pi} \right\rangle.$$

(a) Show that the graphs of the given vector-functions intersect at the origin.

It is enough to show that $(0,0,0)$ lies on the graphs of $\vec{r}_1(t)$ and $\vec{r}_2(t)$

We need to find t_1 such that $\vec{r}_1(t_1) = <0,0,0>$

t_2 such that $\vec{r}_2(t_2) = <0,0,0>$

$<\arctan t_1, t_1, -t_1^4> = <0,0,0>$

$t_1 = 0$

$< t_2^2 - t_2, 2\ln t_2, \frac{\sin(2\pi t_2)}{2\pi}> = <0,0,0>$

$t_2 = 1$

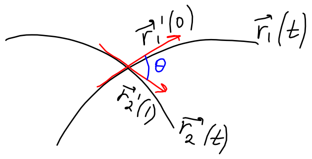
we've showed that $(0,0,0)$ lies on both $\vec{r}_1(t)$ and $\vec{r}_2(t)$.

(b) Find their angle of intersection at the origin.

$\vec{r}_1{}'(t) = < \frac{1}{1+t^2}, 1, -4t^3>$, $\vec{r}_1{}'(0) = <1,1,0>$

$\vec{r}_2{}'(t) = <2t-1, \frac{2}{t}, \cos(2\pi t)>$

$\vec{r}_2{}'(1) = <1, 2, 1>$



$$\cos\theta = \frac{\vec{r}_1{}'(0)\cdot\vec{r}_2{}'(1)}{|\vec{r}_1{}'(0)|\cdot|\vec{r}_2{}'(1)|} = \frac{3}{\sqrt{1+1}\sqrt{1+4+1}} = \frac{3}{\sqrt{12}} = \frac{\sqrt{3}}{2}$$

$$\boxed{\theta = \pi/6}$$

12. Evaluate the integral $\int_1^4 \left(\sqrt{t}\vec{\imath} + te^{-t}\vec{\jmath} + \frac{1}{t^2}\vec{k}\right) dt$

$$= \left(\int_1^4 \sqrt{t}\, dt\right)\vec{\imath} + \left(\int_1^4 t e^{-t} dt\right)\vec{\jmath} + \left(\int_1^4 \frac{1}{t^2} dt\right)\vec{k}$$

13. A moving particle starts at an initial position $\vec{r}(0) =< 1, 0, 0 >$ with initial velocity $\vec{v}(0) = \vec{\imath} - \vec{\jmath} + \vec{k}$. Its acceleration is $\vec{a}(t) = 4t\vec{\imath} + 6t\vec{\jmath} + \vec{k}$. Find its velocity and position at time t.