$f(x,y)$ Range is the set of all values that f takes on.
Domain is the set of all ordered pairs (x,y) such the equation $f(x,y)$ makes sense.

Math 251. WEEK in REVIEW 3. Fall 2013

1. Let $f(x,y)=\sin(xy)+\pi$. Find $f(x-y, x+y)$.

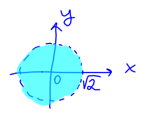
$f(x-y, x+y) = \sin\left((x-y)(x+y)\right)+\pi = \boxed{\sin(x^2-y^2)+\pi}$

2. Let $f(x,y,z)=y+xz$. Find $f(y-z, x^3, y+z)$.

$f(y-z, x^3, y+z) = x^3+(y-z)(y+z)$
$= \boxed{x^3+y^2-z^2}$

3. Find the domain and range of the functions:

(a) $f(x,y)=\ln(2-x^2-y^2)$;



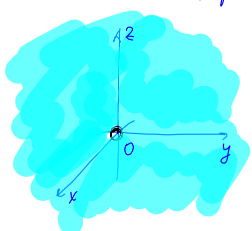
$\boxed{\text{Domain: } 2-x^2-y^2>0,\ x^2+y^2<2}$ inside of a circle of radius $\sqrt{2}$ with center at $(0,0)$

Range: $0<x^2+y^2<2$
$\cancel{\ln 0}$ $-\infty$ $<\ln(x^2+y^2)<\ln 2$
$\boxed{\text{Range: } (-\infty, \ln 2)}$

(b) $f(x,y,z)=e^{-\frac{1}{x^2+y^2+z^2}}$

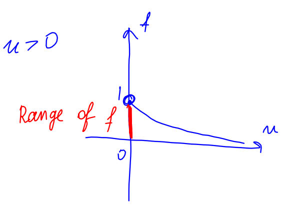
not defined at $(0,0,0)$.

$\boxed{\text{Domain}: \{(x,y,z) \mid (x,y,z)\neq(0,0,0)\}}$



Range.
$u = +\frac{1}{x^2+y^2+z^2}$, $u>0$
$f=e^{-u}$

$\boxed{\text{Range of } f \text{ is } (0,1)}$



a graph of a function $f(x,y)$ is a surface defined by the equation $z=f(x,y)$

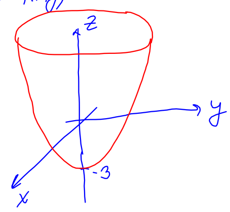
4. Sketch the graphs of the functions:

(a) $f(x, y) = x^2 + y^2 - 3$

$z = x^2+y^2-3$

elliptic paraboloid

vertex $(0,0,-3)$

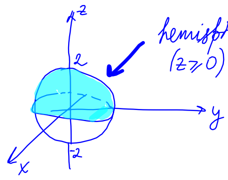


(b) $f(x, y) = \sqrt{4 - x^2 - y^2}$

$(z)^2 = \left(\sqrt{4-x^2-y^2}\right)^2, \quad z \geq 0$

$z^2 = 4-x^2-y^2$

$x^2+y^2+z^2=4$ sphere of radius 2 centered at $(0,0,0)$

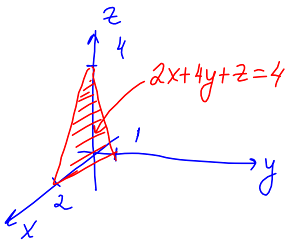


hemisphere $(z \geq 0)$

(c) $f(x, y) = -2x - 4y + 4$

$z = -2x-4y+4$

$2x+4y+z=4$ plane



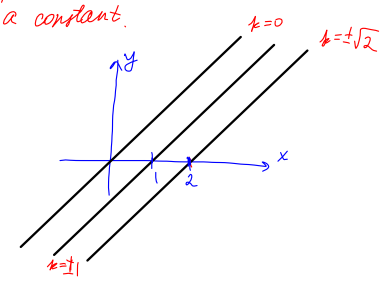
level curve of a function $f(x,y)$ is a curve with an equation $f(x,y)=k$, k is a constant.

5. Draw several level curves of the functions

(a) $f(x,y)=\sqrt{x-y}$

$(\sqrt{x-y})^2 = k^2$, k is a constant

$x-y = k^2$

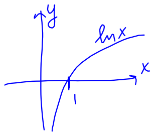


(b) $f(x,y)=e^{-x^2-y^2}$

$\ln(e^{-x^2-y^2}) = \ln(k)$

$-x^2-y^2 = \ln k$

$x^2+y^2 = -\ln k = \ln \frac{1}{k}$

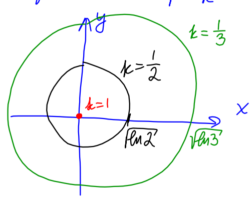


$x^2+y^2 = \ln\frac{1}{k}$ circles centered at 0 of radius $\sqrt{\ln\frac{1}{k}}$

$x^2+y^2 \geq 0$

$\ln\frac{1}{k} > 0$

$0 < k \leq 1$

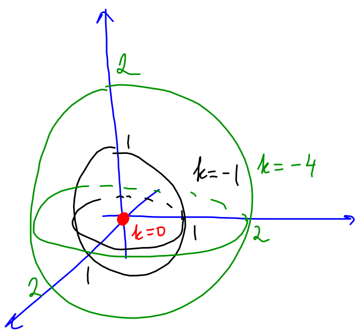


For a function $f(x,y,z)$, lever surfaces are defined by $k=f(x,y,z)$

6. Describe and sketch several level surfaces of the function $f(x,y,z)=-x^2-y^2-z^2$

$k = -x^2-y^2-z^2$

$x^2+y^2+z^2 = -k$, $k \leq 0$



Partial derivative: $\frac{\partial f}{\partial x} = \boxed{f_x}$ of f WRT x. $f(x,y)$ (you differentiate for x and treat y as a constant).

7. Find the first partial derivatives of the functions

(a) $f(x, y) = y^{x^y}$ $\quad (a^x)' = a^x \ln a \quad (x^n)' = n x^{n-1}$

$$\frac{\partial f}{\partial x} = f_x = y^{x^y} \cdot \ln y \cdot (x^y)_x'$$

$$= \boxed{y^{x^y} \ln y \; y x^{y-1} = f_x}$$

$$\ln(f) = \ln(y^{x^y})$$

$$\ln f = x^y \ln y$$

$$\frac{\partial}{\partial y}(\ln f) = \frac{\partial}{\partial y}(x^y \ln y)$$

$$\frac{1}{f}\frac{\partial f}{\partial y} = (x^y)_y' \ln y + x^y (\ln y)_y'$$

$$\frac{1}{f}\frac{\partial f}{\partial y} = x^y \ln x \ln y + \frac{x^y}{y}$$

$$\frac{\partial f}{\partial y} = f\left(x^y \ln x \ln y + \frac{x^y}{y}\right)$$

$$\boxed{f_y = y^{x^y}\left(x^y \ln x \ln y + \frac{x^y}{y}\right)}$$

(b) $g(x, y, z) = x \cos \dfrac{y}{x+z}$

$$\frac{\partial g}{\partial x} = \cos\frac{y}{x+z} + x\left(-\sin\frac{y}{x+z}\right) \cdot \frac{\partial}{\partial x}\left(\frac{y}{x+z}\right)$$

$$= \cos\frac{y}{x+z} + x\left(-\sin\frac{y}{x+z}\right)\frac{-y}{(x+z)^2} = \boxed{\cos\frac{y}{x+z} + \frac{xy}{(x+z)^2}\sin\left(\frac{y}{x+z}\right)}$$

$$\frac{\partial g}{\partial y} = x\left(-\sin\frac{y}{x+z}\right)\frac{\partial}{\partial y}\left(\frac{y}{x+z}\right)$$

$$= \boxed{-x \sin\frac{y}{x+z} \cdot \frac{1}{x+z}}$$

$$\frac{\partial g}{\partial z} = x\left(-\sin\frac{y}{x+z}\right)\frac{\partial}{\partial z}\left(\frac{y}{x+z}\right)$$

$$= \boxed{x\left(-\sin\frac{y}{x+z}\right)\left(-\frac{y}{(x+z)^2}\right)}$$

(c) $h(x, y, z) = xz \tan\left(\dfrac{y}{x}\right)$

$$\boxed{h_z = x \tan\left(\frac{y}{x}\right)}$$

$$h_y = xz \sec^2\left(\frac{y}{x}\right)\frac{\partial}{\partial y}\left(\frac{y}{x}\right) \quad 1/x$$

$$= \cancel{x}z \sec^2\left(\frac{y}{x}\right) \cdot \frac{1}{\cancel{x}} = \boxed{z \sec^2\left(\frac{y}{x}\right)}$$

Product Rule

$$h_x = z \tan\left(\frac{y}{x}\right) + xz \sec^2\left(\frac{y}{x}\right)\frac{\partial}{\partial x}\left(\frac{y}{x}\right) \quad -\frac{y}{x^2}$$

$$= z \tan\left(\frac{y}{x}\right) + \cancel{x}z \sec^2\left(\frac{y}{x}\right)\left(-\frac{y}{x^{\cancel{2}}}\right)$$

$$= \boxed{z \tan\left(\frac{y}{x}\right) - \frac{yz}{x}\sec^2\left(\frac{y}{x}\right)}$$

8. Find $\dfrac{\partial z}{\partial x}$, $\dfrac{\partial z}{\partial y}$ if

$$xyz = \cos(x + y + z)$$

implicit differentiation

$$z = z(x,y)$$

$$(xyz)_x' = (\cos(x+y+z))_x'$$

product Rule

$$yz + xy\, z_x = -\sin(x+y+z)\,(x+y+z)_x' \quad 1 + z_x$$

$$yz + xy\, z_x = -(1+z_x)\sin(x+y+z)$$

$$yz + xy\, z_x = -\sin(x+y+z) - z_x \sin(x+y+z)$$

$$z_x\left(xy + \sin(x+y+z)\right) = -\sin(x+y+z) - yz$$

$$\boxed{z_x = \frac{-\sin(x+y+z) - yz}{xy + \sin(x+y+z)}}$$

z_y: $\quad (xyz)_y' = (\cos(x+y+z))_y'$

$$xz + xy\, z_y = -\sin(x+y+z)\,(x+y+z)_y' \quad 1 + z_y$$

$$xz + xy\, z_y = -(1+z_y)\sin(x+y+z)$$

solve for z_y

$$\boxed{z_y = \frac{-\sin(x+y+z) - xz}{xy + \sin(x+y+z)}}$$

Second order derivatives: $f_{xx}=(f_x)_x$, $f_{yy}=(f_y)_y$, $f_{xy}=(f_x)_y=(f_y)_x$

9. Determine whether the function

$$f(x, y) = e^{-x}\cos y - e^{-y}\cos x$$

satisfies the Laplace equation $f_{xx} + f_{yy} = 0$.

$f_x = -e^{-x}\cos y - e^{-y}(-\sin x)$
$= -e^{-x}\cos y + e^{-y}\sin x$

$f_{xx} = (-e^{-x}\cos y + e^{-y}\sin x)'_x$
$= e^{-x}\cos y + e^{-y}\cos x$

$f_y = -e^{-x}\sin y + e^{-y}\cos x$

$f_{yy} = (-e^{-x}\sin y + e^{-y}\cos x)'_y$
$= -e^{-x}\cos y - e^{-y}\cos x$

$f_{xx} + f_{yy} = \cancel{e^{-x}\cos y} + \cancel{e^{-y}\cos x} + (\cancel{-e^{-x}\cos y} - \cancel{e^{-y}\cos x})$
$= 0$

$\boxed{\text{YES}}$

10. Determine whether the function

$$f(t, x) = \cos(x - at) + (x + at)^2$$

satisfies the wave equation $f_{tt} = a^2 f_{xx}$,

$f_t = -\sin(x-at)(-a) + 2(x+at)a$ $(x-at)'_t$ $(x+at)'_t$
$= a\sin(x-at) + 2ax + 2a^2t$

$f_{tt} = a\cos(x-at)(-a) + 2a^2$
$= -a^2\cos(x-at) + 2a^2$
$= a^2(-\cos(x-at) + 2)$
$= a^2 f_{xx}$

$f_x = -\sin(x-at) + 2(x+at)$
$= -\sin(x-at) + 2x + 2at$

$f_{xx} = -\cos(x-at) + 2$

$\boxed{\text{YES}}$

11. If $u = x^y$, show that $\frac{x}{y}\frac{\partial u}{\partial x} + \frac{1}{\ln x}\frac{\partial u}{\partial y} = 2u$.

$(x^n)'$ $\quad$ $(a^x)'$

$u = x^y$

$u_x = y\, x^{y-1}$ $\qquad$ $u_y = x^y \ln x$

$\frac{x}{\cancel{y}}(\cancel{y}\, x^{y-1}) + \frac{1}{\cancel{\ln x}}(x^y \cancel{\ln x})$

$= x\, x^{y-1} + x^y$

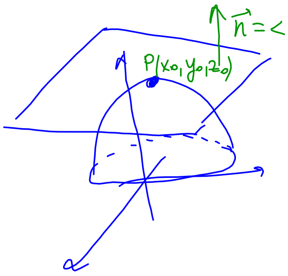
$= x^y + x^y$

$= 2x^y$

$= 2u$

12. Where does the plane tangent to the surface $z = e^{x-y}$ at $(1, 1, 1)$ meet the z-axis?

$\vec{n} = \langle z_x(x_0,y_0), z_y(x_0,y_0), -1 \rangle$



The equation of the tangent plane is:

$\boxed{z_x(x_0,y_0)(x-x_0) + z_y(x_0,y_0)(y-y_0) - (z-z_0) = 0}$

$z = e^{x-y}$ at $(1,1,1)$ $\quad$ $x_0=1,\ y_0=1,\ z_0=1$

$z_x = e^{x-y}$ $\qquad$ $z_x(1,1) = e^{1-1} = e^0 = 1$

$z_y = -e^{x-y}$ $\qquad$ $z_y(1,1) = -e^{1-1} = -e^0 = -1$

Tangent plane: $\quad 1(x-1) + (-1)(y-1) - (z-1) = 0$

$x - 1 - y + 1 - z + 1 = 0$

$\boxed{x - y - z + 1 = 0}$

z-intersept. plug $x = y = 0$ $\qquad$ $-z + 1 = 0$

$\boxed{z = 1}$

z-intersept is $\boxed{(0,0,1)}$

13. Show that the surfaces given by $f(x,y) = x^2 + y^2$ and $g(x,y) = -x^2 - y^2 + xy^3$ have the same tangent plane at $(0,0)$. $x_0 = y_0 = 0$

$f(x,y) = x^2 + y^2$

$z_0 = f(x_0, y_0) = f(0,0) = 0$

tangent plane at $(0,0,0)$

$\vec{n} = < f_x(x_0,y_0), f_y(x_0,y_0), -1 >$

$f_x = 2x \qquad f_x(0,0) = 0$

$f_y = 2y \qquad f_y(0,0) = 0$

$\vec{n}_1 = <0, 0, -1>$

Tangent plane:

$0(x-0) + 0(y-0) - 1(z-0) = 0$

$\boxed{z=0}$

$g(x,y) = -x^2 - y^2 + xy^3$

$z_0 = g(0,0) = 0$

tangent plane at $(0,0,0)$

$\vec{n}_2 = < g_x(0,0), g_y(0,0), -1 >$

$g_x = -2x + y^3 \qquad g_x(0,0) = 0$

$g_y = -2y + 2y^2x \qquad g_y(0,0) = 0$

$\vec{n}_2 = < 0, 0, -1 >$

Tangent plane:

$0(x-0) + 0(y-0) - 1(z-0) = 0$

$\boxed{z=0}$

14. Find the differential of the function

$$f(x,y,z) = \sqrt{x^2 + y^2 + z^2}$$

Differential of a function $f(x,y,z)$

$df = f_x(x,y,z)\,dx + f_y(x,y,z)\,dy + f_z(x,y,z)\,dz$

$f_x = \frac{1}{\cancel{2}}(x^2+y^2+z^2)^{-1/2}(\cancel{2}x)$

$f_y = \frac{1}{\cancel{2}}(x^2+y^2+z^2)^{-1/2}(\cancel{2}y)$

$f_z = \frac{1}{\cancel{2}}(x^2+y^2+z^2)^{-1/2}(\cancel{2}z)$

$\boxed{df = x(x^2+y^2+z^2)^{-1/2}dx + y(x^2+y^2+z^2)^{-1/2}dy + z(x^2+y^2+z^2)^{-1/2}dz}$

$$\boxed{f(a+\Delta x, b+\Delta y, c+\Delta z) \approx f(a,b,c) + f_x(a,b,c)\Delta x + f_y(a,b,c)\Delta y + f_z(a,b,c)\Delta z}$$

15. Use differentials to estimate

$$\sqrt{(4.01)^2 + (3.98)^2 + (2.02)^2}$$

$f(x,y,z) = \sqrt{x^2+y^2+z^2}$, $f(a,b,c) = f(4,4,2) = \sqrt{4^2+4^2+2^2} = 6$

$a+\Delta x = 4.01$ $\quad a=4,\ \Delta x = 0.01$

$b+\Delta y = 3.98$ $\quad b=4,\ \Delta y = -0.02$

$c+\Delta z = 2.02$ $\quad c=2,\ \Delta z = 0.02$

$f_x = x(x^2+y^2+z^2)^{-1/2}$ $\quad \frac{4}{6} = \frac{2}{3} = f_x(4,4,2)$

$f_y = y(x^2+y^2+z^2)^{-1/2}$ $\quad \frac{4}{6} = \frac{2}{3} = f_y(4,4,2)$

$f_z = z(x^2+y^2+z^2)^{-1/2}$ $\quad \frac{2}{6} = \frac{1}{3} = f_z(4,4,2)$

$$\sqrt{(4.01)^2+(3.98)^2+(2.02)^2} \approx f(4,4,2) + f_x(4,4,2)\Delta x + f_y(4,4,2)\Delta y + f_z(4,4,2)\Delta z$$

$$= \boxed{6 + \frac{2}{3}(0.01) - \frac{2}{3}(0.02) + \frac{1}{3}(0.02)}$$

16. The two legs of a right triangle are measured as 5 m and 12 m respectively, with a possible error in measurement of at most 0.2 cm in each. Use differentials to estimate the maximum error in the calculated value of the area of the triangle.

$A = \frac{1}{2}xy$, $\quad a=5,\ b=12,\ \Delta x = 0.2,\ \Delta y = 0.2$

Error:

$$\boxed{\Delta A(a,b) \approx A_x(a,b)\Delta x + A_y(a,b)\Delta y}$$

$A_x = \frac{1}{2}y,\ A_y = \frac{1}{2}x$, $\quad A_x(5,12) = \frac{12}{2} = 6$

$A_y(5,12) = \frac{5}{2}$

$$\boxed{\Delta A(5,12) \approx 6(0.2) + \frac{5}{2}(0.2)}$$