# Math 251. WEEK in REVIEW 5. Fall 2013

1. Given $\vec{a} = < 1, 1, 2 >$ and $\vec{b} = < 2, -1, 0 >$. Find the area of the parallelogram with adjacent sides $\vec{a}$ and $\vec{b}$.

SOLUTION. $A = |\vec{a} \times \vec{b}|$

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ 1 & 1 & 2 \\ 2 & -1 & 0 \end{vmatrix} = \vec{\imath} \begin{vmatrix} 1 & 2 \\ -1 & 0 \end{vmatrix} - \vec{\jmath} \begin{vmatrix} 1 & 2 \\ 2 & 0 \end{vmatrix} + \vec{k} \begin{vmatrix} 1 & 1 \\ 2 & -1 \end{vmatrix} = 2\vec{\imath} + 4\vec{\jmath} - 3\vec{k}$$

$$|\vec{a} \times \vec{b}| = \sqrt{(2)^2 + (4)^2 + (-3)^2} = \sqrt{29}$$

Thus, $A = \sqrt{29}$.

2. Find an equation of the line through the point $(1, 2, -1)$ and perpendicular to the plane

$$2x + y + z = 2$$

SOLUTION. The line is parallel to the normal vector of the plane $\vec{n} = < 2, 1, 1 >$. Thus, symmetric equations of the line are:

$$\frac{x-1}{2} = \frac{y-2}{1} = \frac{z+1}{1}$$

3. Find the distance from the point $(1, -1, 2)$ to the plane

$$x + 3y + z = 7$$

SOLUTION. $D = \dfrac{|1 + 3(-1) + 2 - 7|}{\sqrt{(1)^2 + (3)^2 + (1)^2}} = \dfrac{7}{\sqrt{11}}$.

4. Find an equation of the plane that passes through the point $(-1, -3, 1)$ and contains the line $x = -1 - 2t$, $y = 4t$, $z = 2 + t$.

SOLUTION. The vector $\vec{v} = < -2, 4, 1 >$ lies in the plane. Let $P(-1, -3, 1)$ and $Q(-1, 0, 2)$. The second vector that lies in the plane is the vector $\overrightarrow{PQ} = < 0, 3, 1 >$. Then the normal vector to the plane

$$\vec{n} = \vec{v} \times \overrightarrow{PQ} = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ -2 & 4 & 1 \\ 0 & 3 & 1 \end{vmatrix} = \vec{\imath} \begin{vmatrix} 4 & 1 \\ 3 & 1 \end{vmatrix} - \vec{\jmath} \begin{vmatrix} -2 & 1 \\ 0 & 1 \end{vmatrix} + \vec{k} \begin{vmatrix} -2 & 4 \\ 0 & 3 \end{vmatrix} = \vec{\imath} + 2\vec{\jmath} - 6\vec{k}$$

Thus, an equation of the plane is

$$1(x+1) + 2(y+3) - 6(z-1) = 0$$

5. Find parametric equations of the line of intersection of the planes $z = x + y$ and $2x - 5y - z = 1$.

SOLUTION. The direction vector for the line of intersection is $\vec{v} = \vec{n_1} \times \vec{n_1}$, where $\vec{n_1} = < 1, 1, -1 >$ is the normal vector for the first plane and $\vec{n_2} = < 2, -5, -1 >$ is the normal vector for the second plane.

$$\vec{v} = \vec{n_1} \times \vec{n_2} = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ 1 & 1 & -1 \\ 2 & -5 & -1 \end{vmatrix} = \vec{\imath}\begin{vmatrix} 1 & -1 \\ -5 & -1 \end{vmatrix} - \vec{\jmath}\begin{vmatrix} 1 & -1 \\ 2 & -1 \end{vmatrix} + \vec{k}\begin{vmatrix} 1 & 1 \\ 2 & -5 \end{vmatrix} =$$

$$\vec{\imath}(-1-5) - \vec{\jmath}(-1+2) + \vec{k}(-5-2) = -6\vec{\imath} - \vec{\jmath} - 7\vec{k}$$

. To find a point on the line of intersection, set one of the variables equal to a constant, say $y = 0$. Then the equations of the planes reduce to $x - z = 0$ and $2x - z = 1$. Solving this two equations gives $x = z = 1$. So a point on a line of intersection is $(1, 0, 1)$. The parametric equations for the line are

$$x = 1 - 6t$$
$$y = -t$$
$$z = 1 - 7t$$

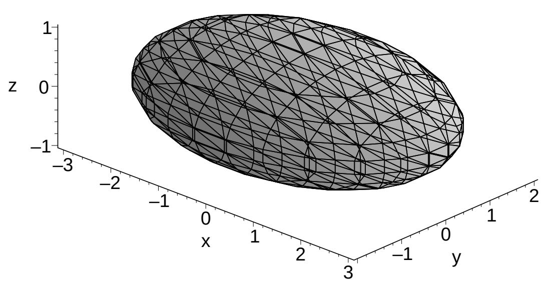
6. Are the lines $x = -1 + 4t$, $y = 3 + t$, $z = 1$ and $x = 13 - 8s$, $y = 1 - 2s$, $z = 2$ parallel, skew or intersecting? If they intersect, find the point of intersection.

   SOLUTION. The direction vector for the first line is $\vec{v_1} =< 4, 1, 0 >$, the second line is parallel to the vector $\vec{v_2} =< -8, -2, 0 >$. Since $\vec{v_2} = -2\vec{v_1}$, vectors $\vec{v_1}$ and $\vec{v_2}$ are parallel. Thus, the lines are parallel.

7. Identify and roughly sketch the following surfaces. Find traces in the planes $x = k$, $y = k$, $z = k$

   (a) $4x^2 + 9y^2 + 36z^2 = 36$

   SOLUTION. $\dfrac{x^2}{9} + \dfrac{y^2}{4} + z^2 = 1$ - ellipsoid




Traces

in $x = k$: $\dfrac{y^2}{4(1-\frac{k^2}{9})} + \dfrac{z^2}{1-\frac{k^2}{9}} = 1$ - ellipse

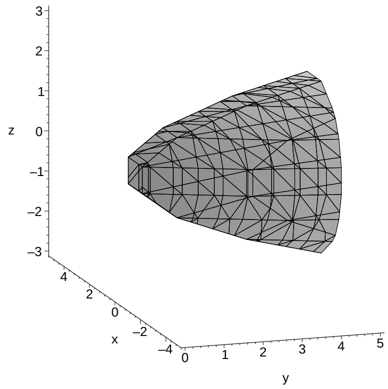
in $y = k$: $\dfrac{x^2}{9(1-\frac{k^2}{4})} + \dfrac{z^2}{1-\frac{k^2}{4}} = 1$ - ellipse

in $z = k$: $\dfrac{x^2}{9(1-k^2)} + \dfrac{y^2}{4(1-k^2)} = 1$ - ellipse

   (b) $y = x^2 + z^2$

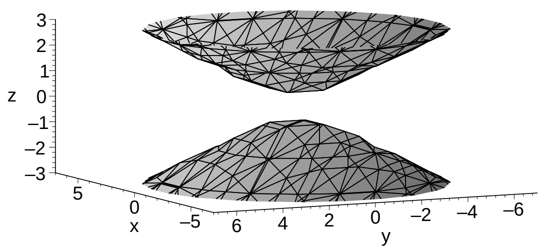
   An equation $y = x^2 + z^2$ defines the elliptic paraboloid with axis the $y$-axis.

   SOLUTION.



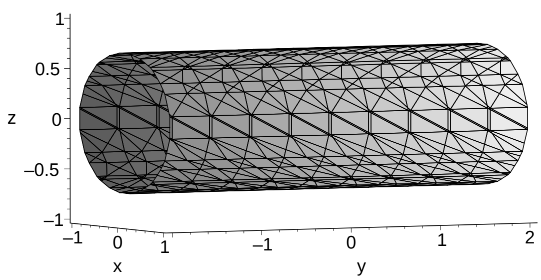

Traces
in $x = k$: $y = z^2 + k^2$ - parabola
in $y = k$: $x^2 + z^2 = k$ - circle
in $z = k$: $y = x^2 + k^2$ - parabola

(c) $4z^2 - x^2 - y^2 = 1$
SOLUTION. An equation $4z^2 - x^2 - y^2 = 1$ defines the hyperboloid on two sheets with axis the $z$-axis




Traces
in $x = k$: $\dfrac{z^2}{1/4(1-k^2)} - \dfrac{x^2}{1-k^2} = 1$ - hyperbola
in $y = k$: $\dfrac{z^2}{1/4(1-k^2)} - \dfrac{y^2}{1-k^2} = 1$ - hyperbola
in $z = k$ ($|k| > 1/2$): $x^2 + y^2 = 4k^2 - 1$ - circle

(d) $x^2 + 2z^2 = 1$
SOLUTION. An equation $x^2 + 2z^2 = 1$ defines the elliptic cylinder with axis $y$-axis.




Traces
in $x = k$: $z = \sqrt{\dfrac{1-k^2}{2}}$, $z = -\sqrt{\dfrac{1-k^2}{2}}$ - lines
in $z = k$: $x = \sqrt{1-2k^2}$, $x = -\sqrt{1-2k^2}$ - lines

8. Find

$$\lim_{t\to 1}\left(\sqrt{t+3}\vec{\imath} + \frac{t-1}{t^2-1}\vec{\jmath} + \frac{\tan t}{t}\vec{k}\right)$$

SOLUTION.

$$\lim_{t\to 1}\left(\sqrt{t+3}\vec{\imath} + \frac{t-1}{t^2-1}\vec{\jmath} + \frac{\tan t}{t}\vec{k}\right) = \lim_{t\to 1}\sqrt{t+3}\vec{\imath} + \lim_{t\to 1}\frac{t-1}{t^2-1}\vec{\jmath} + \lim_{t\to 1}\frac{\tan t}{t}\vec{k} =$$

$$\sqrt{4}\vec{\imath} + \lim_{t\to 1}\frac{t-1}{(t-1)(t+1)}\vec{\jmath} + \tan(1)\vec{k} = 2\vec{\imath} + \frac{1}{2}\vec{\jmath} + \tan(1)\vec{k}$$

9. Find the unit tangent vector $\vec{T}(t)$ for the vector function $\vec{r}(t) = < t, 2\sin t, 3\cos t >$.

SOLUTION. The tangent vector $\vec{r'}(t) =< 1, 2\cos t, -3\sin t >$,

$$|\vec{r'}(t)| = \sqrt{1 + 4\cos^2 t + 9\sin^2 t} = \sqrt{1 + 4\cos^2 t + 4\sin^2 t + 5\sin^2 t} = \sqrt{5 + 5\sin^2 t}$$

. The unit tangent vector

$$\vec{T}(t) = \frac{1}{|\vec{r'}(t)|}\vec{r'}(t) = \frac{1}{\sqrt{5 + 5\sin^2 t}} < t, 2\cos t, -3\sin t >$$

10. Evaluate

$$\int_1^4 \left(\sqrt{t}\vec{\imath} + te^{-t}\vec{\jmath} + \frac{1}{t^2}\vec{k}\right) dt$$

SOLUTION.

$$\int_1^4 \left(\sqrt{t}\vec{\imath} + te^{-t}\vec{\jmath} + \frac{1}{t^2}\vec{k}\right) dt = \left(\int_1^4 \sqrt{t}dt\right)\vec{\imath} + \left(\int_1^4 te^{-t}dt\right)\vec{\jmath} + \left(\int_1^4 \frac{1}{t^2}dt\right)\vec{k}$$

$$\int_1^4 \sqrt{t}dt = \left.\frac{t^{3/2}}{3/2}\right|_1^4 = \frac{2}{3}[4^{3/2} - 1] = \frac{2}{3}(8 - 1) = \frac{14}{3}$$

$$\int_1^4 te^{-t}dt = \left| \begin{array}{ll} u = t & u' = 1 \\ v' = e^{-t} & v = -e^{-t} \end{array} \right| = -te^{-t}|_1^4 + \int_1^4 e^{-t}dt = -4e^{-4} + e^{-1} - e^{-t}|_1^4 =$$

$$-4e^{-4} + e^{-1} - e^{-4} + e^{-1} = -5e^{-4} + 2e^{-1}$$

$$\int_1^4 \frac{1}{t^2}dt = \left.\frac{1}{t}\right|_1^4 = -\frac{1}{4} + 1 = \frac{3}{4}$$

Thus,

$$\int_1^4 \left(\sqrt{t}\vec{\imath} + te^{-t}\vec{\jmath} + \frac{1}{t^2}\vec{k}\right) dt = \frac{14}{3}\vec{\imath} + (-5e^{-4} + 2e^{-1})\vec{\jmath} + \frac{3}{4}\vec{k}$$

11. Find the length of the curve given by the vector function $\vec{r}(t) = \cos^3 t\ \vec{\imath} + \sin^3 t\ \vec{\jmath} + \cos(2t)\ \vec{k}$, $0 \le t \le \frac{\pi}{2}$.

SOLUTION. $\vec{r'}(t) = -3\cos^2 t \sin t\ \vec{\imath} + 3\sin^2 t \cos t\ \vec{\jmath} - 2\sin(2t)\ \vec{k}$

$|\vec{r'}(t)| = \sqrt{9\cos^4 t \sin^2 t\ + 9\sin^4 t \cos^2 t\ + 4\sin^2(2t)}$

Recall that $\sin(2t) = 2\sin t\cos t$, then $\sin^2(2t) = 4\sin^2 t\cos^2 t$ and

$|\vec{r'}(t)| = \sqrt{\sin^2 t\cos^2 t(9\sin^2 t + 9\cos^2 t + 16)} = \sin t\cos t\sqrt{25} = 5\sin t\cos t = \frac{5}{2}\sin(2t)$

Then the length of the curve

$L = \int_0^{\pi/2} \frac{5}{2}\sin(2t)\ dt = -\frac{5}{4}\cos(2t)\Big|_0^{\pi/2} = \frac{5}{2}$

12. Find the curvature of the curve $\vec{r}(t) = < 2t^3, -3t^2, 6t >$.

SOLUTION.

$$\kappa(t) = \frac{|\vec{r'}(t) \times \vec{r''}(t)|}{|\vec{r'}(t)|^3}$$

$\vec{r'}(t) = < 6t^2, -6t, 6 > = 6 < t^2, -t, 1 >$, $|\vec{r'}(t)| = 6\sqrt{1 + t^2 + t^4}$

$\vec{r''}(t) = < 12t, -6, 0 >$

$$\vec{r'}(t) \times \vec{r''}(t) = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ 6t^2 & -6t & 6 \\ 12t & -6 & 0 \end{vmatrix} = \vec{\imath} \begin{vmatrix} -6t & 6 \\ -6 & 0 \end{vmatrix} - \vec{\jmath} \begin{vmatrix} 6t^2 & 6 \\ 12t & 0 \end{vmatrix} + \vec{k} \begin{vmatrix} 6t^2 & -6t \\ 12t & -6 \end{vmatrix} =$$

$$\vec{\imath}(36) - \vec{\jmath}(-72t) + \vec{k}(-36t^2 + 72t^2) = < 36, 72t, 36t^2 > = 36 < 1, 2t, t^2 >$$

$|\vec{r'}(t) \times \vec{r''}(t)| = 36\sqrt{1 + 4t^2 + t^4}$

Thus,

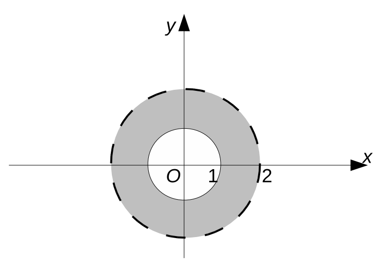
$$\kappa(t) = \frac{36\sqrt{1 + 4t^2 + t^4}}{(6\sqrt{1 + t^2 + t^4})^3} = \frac{\sqrt{1 + 4t^2 + t^4}}{6(1 + t^2 + t^4)^{3/2}}$$

13. Sketch the domain of the function

$$f(x, y) = \sqrt{x^2 + y^2 - 1} + \ln(4 - x^2 - y^2)$$

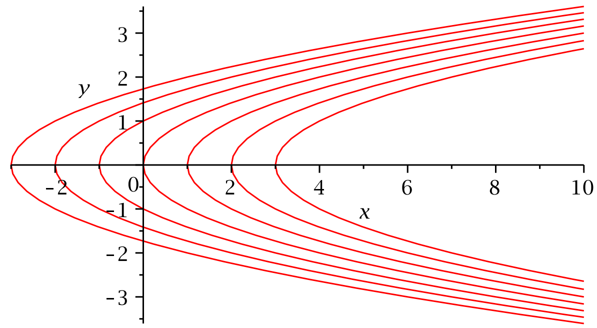
SOLUTION. The expression for $f$ makes sense if $x^2 + y^2 - 1 \geq 0$ and $4 - x^2 - y^2 > 0$. Thus, the domain of the function $f$ is

$$D(f) = \{(x, y) \in \mathbb{R}^2 | 1 \leq x^2 + y^2 < 4\}$$




14. Find the level curves of the function $z = x - y^2$.

SOLUTION. An equation for the level curves is $k = x - y^2$ or $y^2 = x - k$. It defines the family of parabolas.




15. Find $f_{xyz}$ if $f(x, y, z) = e^{xyz}$.

    SOLUTION. $f_x = e^{xyz}(xyz)'_x = yze^{xyz}$

    $f_{xy} = (yze^{xyz})'_y = ze^{xyz} + yze^{xyz}(xyz)'_y = ze^{xyz} + xyz^2e^{xyz}$

    $f_{xyz} = (ze^{xyz} + xyz^2e^{xyz})'_z = e^{xyz} + ze^{xyz}(xyz)'_z + 2xyze^{xyz} + xyz^2e^{xyz}(xyz)'_z = e^{xyz} + xyze^{xyz} + 2xyze^{xyz} + x^2y^2z^2e^{xyz} = (1 + 3xyz + x^2y^2z^2)e^{xyz}$

16. The dimensions of a closed rectangular box are 80 cm, 60 cm, and 50 cm with a possible error of 0.2 cm in each dimension. Use differential to estimate the maximum error in surface area of the box.

    SOLUTION. Let $l$, $w$, and $h$ be the length, width, and height, respectively, of the box in centimeters.

    $$\Delta l = \Delta w = \Delta h = 0.2$$

    The surface area of the box

    $$A(l, w, h) = 2(lw + lh + wh)$$

    $$\Delta A = \frac{\partial A}{\partial l}\Delta l + \frac{\partial A}{\partial w}\Delta w + \frac{\partial A}{\partial h}\Delta h$$

    $$\frac{\partial A}{\partial l} = 2w + 2h; \qquad \frac{\partial A}{\partial l}(80, 60, 50) = 220$$

    $$\frac{\partial A}{\partial w} = 2l + 2h; \qquad \frac{\partial A}{\partial w}(80, 60, 50) = 260$$

    $$\frac{\partial A}{\partial h} = 2l + 2w; \qquad \frac{\partial A}{\partial h}(80, 60, 50) = 280$$

    Thus,

    $$\Delta A = (220 + 260 + 280)(0.2) = 152\text{cm}^2$$

17. Find parametric equations of the normal line and an equation of the tangent plane to the surface

    $$x^3 + y^3 + z^3 = 5xyz$$

    at the point $(2, 1, 1)$.

SOLUTION. Let $F(x,y,z) = x^3 + y^3 + z^3 - 5xyz$. Then

$$\nabla F(x,y,z) =< 3x^2 - 5yz, 3y^2 - 5xz, 3z^2 - 5xy >$$

$$\nabla F(2,1,1) =< 7, -7, -7 >$$

The equation of the tangent plane at $(2,1,1)$ is

$$7(x-2) - 7(y-1) - 7(z-1) = 0$$

The parametric equations of the normal line at $(2,1,1)$ are

$$x = 2 + 7t, \qquad y = 1 - 7t, \qquad z = 1 - 7t$$

18. Given that $w(x,y) = 2\ln(3x+5y) + x - 2\tan^{-1} y$, where $x = s - \cot t$, $y = s + \sin^{-1} t$. Find $\dfrac{\partial w}{\partial t}$.

SOLUTION.

$$\frac{\partial w}{\partial t} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial t} = \left(\frac{6}{3x+5y} + 1\right)\csc^2 t + \left(\frac{10}{3x+5y} - \frac{2}{1+y^2}\right)\frac{1}{\sqrt{1-t^2}}$$

19. Let $f(x,y,z) = \ln(2x+3y+6z)$. Find a unit vector in the direction in which $f$ decreases most rapidly at the point $P(-1,-1,1)$ and find the derivative (rate of change) of $f$ in this direction.

SOLUTION. The function $f$ decreases most rapidly in the direction of the vector $-\nabla f(-1,-1,1)$.

$$\nabla f(x,y,z) = \left\langle \frac{2}{2x+3y+6z}, \frac{3}{2x+3y+6z}, \frac{6}{2x+3y+6z} \right\rangle$$

$$\nabla f(-1,-1,1) = \left\langle \frac{2}{-2-3+6}, \frac{3}{-2-3+6}, \frac{6}{-2-3+6} \right\rangle =< 2, 3, 4 >$$

$$|\nabla f(-1,-1,1)| = \sqrt{4+9+36} = \sqrt{49} = 7$$

The unit vector in the direction of the vector $\nabla f(-1,-1,1)$ is

$$\vec{u} = \frac{1}{7} < 2, 3, 4 >= \left\langle \frac{2}{7}, \frac{3}{7}, \frac{6}{7} \right\rangle$$

So, the function $f$ decreases most rapidly in the direction of the vector $-\vec{u} = \left\langle -\frac{2}{7}, -\frac{3}{7}, -\frac{6}{7} \right\rangle$

$$D_{-\vec{u}} f(x,y,z) = \nabla f(x,y,z) \cdot (-\vec{u}) =$$

$$\left\langle \frac{2}{2x+3y+6z}, \frac{3}{2x+3y+6z}, \frac{6}{2x+3y+6z} \right\rangle \cdot \left\langle -\frac{2}{7}, -\frac{3}{7}, -\frac{6}{7} \right\rangle = -\frac{7}{2x+3y+6z}$$

20. Find $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$ if

$$xe^y + yz + ze^x = 0$$

SOLUTION.

$$\frac{\partial z}{\partial x} = -\frac{\dfrac{\partial F}{\partial x}}{\dfrac{\partial F}{\partial z}}, \qquad \frac{\partial z}{\partial y} = -\frac{\dfrac{\partial F}{\partial y}}{\dfrac{\partial F}{\partial z}}$$

where $F(x, y, z) = xe^y + yz + ze^x$

$$\frac{\partial F}{\partial x} = e^y + ze^x$$

$$\frac{\partial F}{\partial y} = xe^y + z$$

$$\frac{\partial F}{\partial z} = y + e^x$$

So $\dfrac{\partial z}{\partial x} = -\dfrac{e^y + ze^x}{y + e^x}$ and $\dfrac{\partial z}{\partial y} = -\dfrac{xe^y + z}{y + e^x}$.

21. Find the local extrema/saddle points for

$$f(x, y) = 2x^2 + y^2 + 2xy + 2x + 2y$$

SOLUTION. We first locate the critical points:

$f_x(x, y) = 2x + y + 1$

$f_y(x, y) = y + x + 1$

Setting these derivatives equal to zero, we get the following system:

$$\begin{cases} 2x + y + 1 = 0 \\ x + y + 1 = 0 \end{cases}$$

Substitute $y = -1 - x$ from the second equation into the first equation:

$2x + (-1 - x) + 1 = 0$

$x = 0,\ y = -1 - x = -1$

The critical point is $(0, -1)$.

Next we calculate the second partial derivatives:

$f_{xx}(x, y) = 4,\ f_{xy}(x, y) = 2,\ f_{yy}(x, y) = 2.$

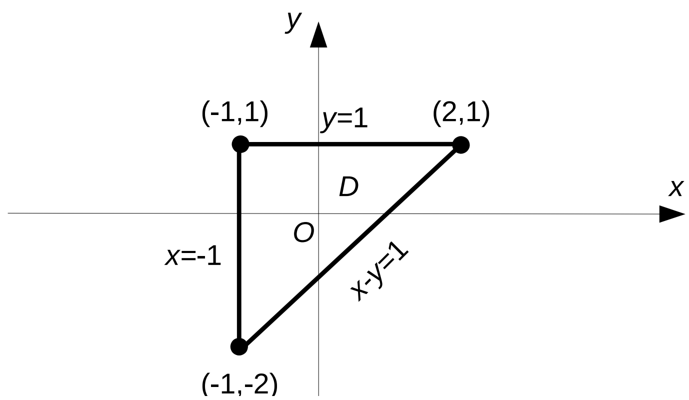
Then

$$D(x, y) = \begin{vmatrix} 4 & 2 \\ 2 & 2 \end{vmatrix} = 8 - 4 = 4 > 0$$

Since $D(x, y) = 4 > 0$ and $f_{xx}(x, y) = 4 > 0$, the function $f$ has a local minimum at the point $(0, -1)$, $f(0, -1) = -1$.

22. Find the absolute maximum and minimum values of the function $f(x, y) = x^2 + 2xy + 3y^2$ over the set $D$, where $D$ is the closed triangular region with vertices $(-1, 1)$, $(2, 1)$, and $(-1, -2)$.

SOLUTION. The set $D$ is bounded by lines $x = -1$, $y = 1$, and $x - y = 1$




First we find critical points for $f$:

$f_x(x, y) = 2x + 2y = 0$

$f_y(x, y) = 2x + 6y = 0$

so the only critical point is $(0, 0)$.

$f(0, 0) = 0$

Now we look at the values of $f$ on the boundary of $D$.

$f(-1, 1) = (-1)^2 + 2(-1)(1) + 3(1)^2 = 2$

$f(2, 1) = (2)^2 + 2(2)(2) + 3(1)^2 = 11$

$f(-1, -2) = (-1)^2 + 2(-1)(-2) + 3(-2)^2 = 17$

If $x = -1$, then $f(-1, y) = 1 - 2y + 3y^2$

$f_y(-1, y) = -2 + 6y = 0$, so $y = 1/3$.

$f(-1, 1/3) = 1/3$

If $y = 1$, then $f(x, 1) = x^2 + 2x + 1$, $f_x(x, 0) = 2x + 2 = 0$ and $x = -1$

$f(-1, 1) = 2$

If $x - y = 1$, then $y = x - 1$, and $g(x) = f(x, y) = x^2 + 2xy + 3y^3 = x^2 + 2x(x-1) + 3(x-1)^2 = 6x^2 - 8x - 3$

$g'(x) = 12x - 8 = 0$, so $x = 8/12 = 2/3$ and $y = 2/3 - 1 = -1/3$

$f(2/3, -1/3) = 1/3$

Thus, the absolute maximum value of the function is $f(-1, -2) = 17$ and the absolute minimum value is $f(0, 0) = 0$.