Math 251. WEEK in REVIEW 6. Fall 2013

1. Find the integral $\iint_R \frac{y\cos y}{x}\,dA$, where $R=\{(x,y)\,|\,1\le x\le e^4, 0\le y\le \pi/2\}$.

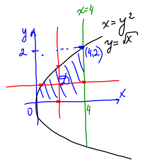
$$\iint_R f(x)g(y)\,dA = \int_a^b f(x)\,dx \int_c^d g(y)\,dy \qquad R=[a,b]\times[c,d]$$

$$\iint_R \frac{y\cos y}{x}\,dA$$
$$= \int_1^{e^4}\int_0^{\pi/2} \frac{y\cos y}{x}\,dy\,dx$$
$$= \int_1^{e^4}\frac{1}{x}\,dx \int_0^{\pi/2} y\cos y\,dy \qquad \left|\begin{array}{ll} u=y & dv=\cos y\,dy \\ du=dy & v=\sin y\end{array}\right|$$

$$\int_a^b u\,dv = uv\Big|_a^b - \int_a^b v\,du$$
integration by parts.

$$= \ln|x|\Big|_1^{e^4}\left(y\sin y\Big|_0^{\pi/2} - \int_0^{\pi/2}\sin y\,dy\right)$$
$$= (\cancelto{4}{\ln e^4} - \cancelto{0}{\ln 1})\left(\frac{\pi}{2}\cancelto{1}{\sin\frac{\pi}{2}} - 0\cancelto{0}{\sin 0} + \cos y\Big|_0^{\pi/2}\right)$$
$$= 4\left(\frac{\pi}{2} - \cancelto{0}{\cos\frac{\pi}{2}} - \cancelto{1}{\cos 0}\right)$$
$$= \boxed{4\left(\frac{\pi}{2}-1\right)}$$

2. Evaluate $\iint_D \frac{y}{\sqrt{1+x^2}}\,dA$ where D is the region in the first quadrant bounded by $x=y^2$, $x=4$, $y=0$.



$0\le y\le\sqrt{x}$
$0\le x\le 4$

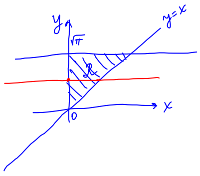
$y^2\le x\le 4$
$0\le y\le 2$

$$\int_0^4\int_0^{\sqrt{x}} \frac{y}{\sqrt{1+x^2}}\,dy\,dx$$
$$= \int_0^2\int_{y^2}^4 \frac{y}{\sqrt{1+x^2}}\,dx\,dy$$
$$\int_0^4 \frac{1}{\sqrt{1+x^2}}\,\frac{y^2}{2}\Big|_{y=0}^{y=\sqrt{x}}\,dx$$
$$= \frac{1}{2\cdot 2}\int_0^4 \frac{2\cdot 1}{\sqrt{1+x^2}}\,x\,dx \qquad \left|\begin{array}{l} 1+x^2=u \\ du=2x\,dx \\ x=0\to u=1 \\ x=4\to u=4^2+1=17\end{array}\right|$$
$$= \frac{1}{4}\int_1^{17}\frac{du}{\sqrt{u}}$$
$$= \frac{1}{4}\,\frac{u^{1/2}}{1/2}\Big|_1^{17}$$
$$= \boxed{\frac{1}{2}\left(\sqrt{17}-1\right)}$$

3. Evaluate $\iint_R y^2\sin\frac{xy}{2}\,dA$ where R is the region bounded by $x=0$ (the y-axis), $y=\sqrt{\pi}$, $y=x$.

$$\int \sin bx\,dx = -\frac{1}{b}\cos bx + C$$

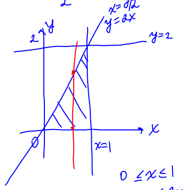
$0\le x\le y$
$0\le y\le\sqrt{\pi}$



$$= \int_0^{\sqrt{\pi}}\int_0^y y^2\sin\frac{xy}{2}\,dx\,dy$$
$$= \int_0^{\sqrt{\pi}} y^2\left(-\cos\frac{xy}{2}\right)\Big|_{x=0}^{x=y}\frac{2}{y}\,dy$$
$$= \int_0^{\sqrt{\pi}} 2y\left[-\cos\frac{y^2}{2} + \cos 0\right]dy$$
$$= 2\int_0^{\sqrt{\pi}} y\left[1-\cos\frac{y^2}{2}\right]dy$$
$$= 2\int_0^{\sqrt{\pi}} y\,dy - 2\int_0^{\sqrt{\pi}} y\cos\frac{y^2}{2}\,dy \qquad \left|\begin{array}{l} u=\frac{y^2}{2} \\ du=y\,dy \\ y=0\to u=0 \\ y=\sqrt{\pi}\to u=\pi/2\end{array}\right|$$
$$= 2\frac{y^2}{2}\Big|_0^{\sqrt{\pi}} - 2\int_0^{\pi/2}\cos u\,du$$
$$= \pi - 2\sin u\Big|_0^{\pi/2}$$
$$= \pi - 2\cancelto{1}{\sin\frac{\pi}{2}} + 2\cancelto{0}{\sin 0}$$
$$= \boxed{\pi-2}$$

4. Evaluate $\int_0^2\int_{y/2}^1 e^{x^2}\,dx\,dy$ by reversing the order of integration.

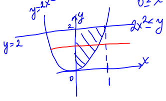
$0\le y\le 2$
$\frac{y}{2}\le x\le 1$



$0\le x\le 1$
$0\le y\le 2x$

$$= \int_0^1\int_0^{2x} e^{x^2}\,dy\,dx$$
$$= \int_0^1 e^{x^2}\,y\Big|_{y=0}^{y=2x}\,dx$$
$$= \int_0^1 e^{x^2}(2x)\,dx \qquad \left|\begin{array}{l} u=x^2 \\ du=2x\,dx \\ x=0\to u=0 \\ x=1\to u=1\end{array}\right|$$
$$= \int_0^1 e^u\,du$$
$$= e^u\Big|_0^1$$
$$= \boxed{e-1}$$

5. Evaluate $\int_0^1 \int_{2x^2}^2 x^3 \sin(y^3)\, dy\, dx$. Reverse the order of integration.



$0 \le x \le 1$
$2x^2 \le y \le 2$

$0 \le y \le 2$
$0 \le x \le \sqrt{\frac{y}{2}}$

$$= \int_0^2 \int_0^{\sqrt{\frac{y}{2}}} x^3 \sin(y^3)\, dx\, dy$$

$$= \int_0^2 \frac{x^4}{4}\Big|_{x=0}^{x=\sqrt{\frac{y}{2}}} \sin(y^3)\, dy$$

$$= \frac{1}{16 \cdot 3}\int_0^2 3y^2 \sin(y^3)\, dy \qquad \left| \begin{array}{l} u=y^3 \\ du = 3y^2 dy \\ y=0 \to u=0 \\ y=2 \to u=8 \end{array} \right|$$

$$= \frac{1}{48}\int_0^8 \sin u\, du$$

$$= -\frac{1}{48} \cos u \Big|_0^8$$

$$= -\frac{1}{48}\left[\cos 8 - \cancel{\cos 0}\right] \quad 1$$

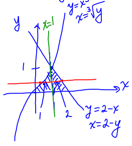
$$= \boxed{-\frac{1}{48}\left[\cos 8 - 1\right]}$$

6. Graph the region and change the order of integration.

a) $\int_0^1 \int_0^{x^3} f(x,y)dy\, dx + \int_1^2 \int_0^{2-x} f(x,y)dydx = \boxed{\int_0^1 \int_{\sqrt[3]{y}}^{2-y} f(x,y)\, dx\, dy}$

$0 \le x \le 1$
$0 \le y \le x^3$
$y = x^3$
$x = \sqrt[3]{y}$

$1 \le x \le 2$
$0 \le y \le 2-x$



$\sqrt[3]{y} \le x \le 2-y$
$0 \le y \le 1$

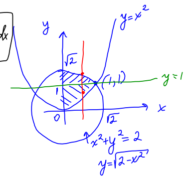
b) $\int_0^1 \int_0^{\sqrt{y}} f(x,y)dxdy + \int_1^{\sqrt{2}} \int_0^{\sqrt{2-y^2}} f(x,y)dxdy = \boxed{\int_0^1 \int_{x^2}^{\sqrt{2-x^2}} f(x,y)\, dy\, dx}$

$0 \le y \le 1$
$0 \le x \le \sqrt{y}$
$x = \sqrt{y}$
$y = x^2$

$1 \le y \le \sqrt{2}$
$0 \le x \le \sqrt{2-y^2}$
$x = \sqrt{2-y^2}$
$x^2 = 2-y^2$
$x^2+y^2 = 2$

Point of intersection
$y = 2-y^2$
$y^2+y-2=0$
$y_1 = 1$, $\cancel{y_2 = -2}$

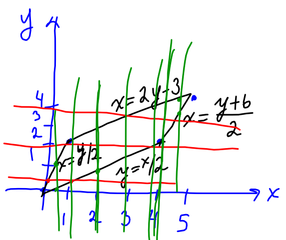
$0 \le x \le 1$
$x^2 \le y \le \sqrt{2-x^2}$



7. Let the region D be the parallelogram with the vertices $(0,0)$, $(1,2)$, $(5,4)$, and $(4,2)$. Write the double integral $\iint_D f(x,y)\, dA$ as a sum of iterated integrals (with the least number of terms).

$$\boxed{\int_0^2 \int_{y/2}^{2y} f(x,y)\, dx\, dy + \int_2^4 \int_{2y-3}^{\frac{y+6}{2}} f(x,y)\, dx\, dy}$$

3 integrals if we do $dy\,dx$.



below $y=2$ $\quad \frac{y}{2} \le x \le 2y$

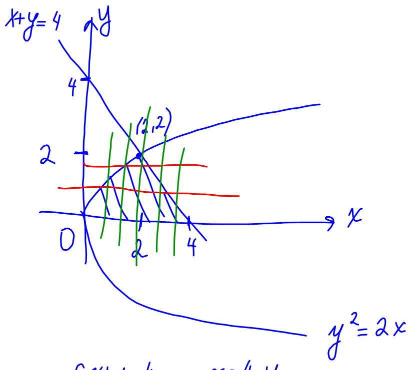
above $y=2$ $\quad 2y-3 \le x \le \frac{y+6}{2}$

$(0,0)$ $(1,2)$ $\quad \vec{v} = \langle 1, 2 \rangle$

$\langle x, y \rangle = \langle 0,0 \rangle + t\langle 1,2 \rangle$

$x = t$
$y = 2t$, $\quad y = 2x$

8. Sketch the region bounded by $y^2 = 2x$ (or $x = \frac{y^2}{2}$), the line $x + y = 4$ and the x-axis, in the first quadrant. Find the area of the region using a double integral.



$$\begin{cases} x+y=4 \longrightarrow x=4-y \\ y^2=2x \end{cases}$$

$$y^2 = 8-2y$$

$$y^2+2y-8=0$$

$$y_1=2,\ \cancel{y_2=-4}$$

$$x=4-y=4-2=2$$

$$A=\iint_{\mathcal{D}} dA$$

$$=\int_0^2\int_{y^2/2}^{4-y} dx\,dy = \int_0^2\int_0^{\sqrt{2x}} dy\,dx+\int_2^4\int_0^{4-x} dy\,dx$$

$$=\int_0^2 x\Big|_{x=\frac{y^2}{2}}^{x=4-y} dy$$

$$=\int_0^2\left(4-y-\frac{y^2}{2}\right)dy$$

$$=\left(4y-\frac{y^2}{2}-\frac{y^3}{6}\right)\Big|_0^2$$

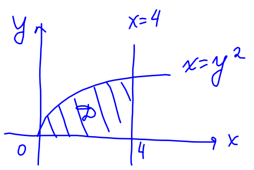
$$=8-2-\frac{8}{6}$$

$$=6-\frac{4}{3}$$

$$=\boxed{\frac{14}{3}}$$

9. Describe the solid which volume is given by the integral $V=\int_0^2\int_{y^2}^4 \overbrace{(x^2+y^2)}^{z=x^2+y^2}dx\,dy$ and find the volume.

Solid that lies above the (xy)-plane and below the elliptic paraboloid $z=x^2+y^2$, above the region $\mathcal{D}$: $0\le y\le 2,\ y^2\le x\le 4$



$$\begin{cases} 0\le z\le x^2+y^2 \\ y^2\le x\le 4 \\ 0\le y\le 2. \end{cases}$$

$$V=\int_0^2\int_{y^2}^4 (x^2+y^2)\,dx\,dy$$

$$=\int_0^2\left(\frac{x^3}{3}+y^2x\right)\Big|_{x=y^2}^{x=4} dy$$

$$=\int_0^2\left(\frac{64}{3}+4y^2-\frac{y^6}{3}-y^4\right)dy$$

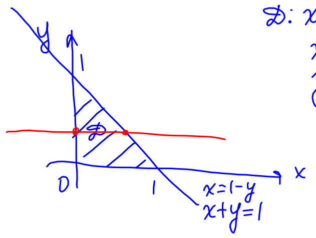
$$=\left(\frac{64}{3}y+\frac{4y^3}{3}-\frac{y^7}{21}-\frac{y^5}{5}\right)\Big|_0^2$$

$$=\boxed{\frac{128}{3}+\frac{32}{3}-\frac{128}{21}-\frac{32}{5}}$$

10. Find the volume of the solid bounded by

$z = 1 + x + y$, $z = 0, x + y = 1, x = 0, y = 0$.

$\mathcal{D}: x+y=1$
$x=0$
$y=0$



$0 \le x \le 1-y$
$0 \le y \le 1$

$$V = \iint_{\mathcal{D}} (1+x+y)\, dA$$

$$= \int_0^1 \int_0^{1-y} (1+x+y)\, dx\, dy = \int_0^1 \int_0^{1-x} (1+x+y)\, dy\, dx$$

$$= \int_0^1 \left(x + \frac{x^2}{2} + xy\right)\Big|_{x=0}^{x=1-y} dy$$

$$= \int_0^1 \left[1 - y + \frac{(1-y)^2}{2} + (1-y)y\right] dy$$

$$= \int_0^1 \left[1 - \cancel{y} + \frac{1}{2}(1 - 2y + y^2) + \cancel{y} - y^2\right] dy$$

$$= \int_0^1 \left[\frac{3}{2} - y - \frac{1}{2}y^2\right] dy$$

$$= \left(\frac{3}{2}y - \frac{y^2}{2} - \frac{y^3}{6}\right)\Big|_0^1$$

$$= \frac{3}{2} - \frac{1}{2} - \frac{1}{6} = \boxed{\frac{5}{6}}$$