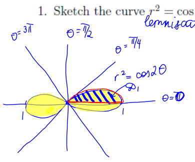
Math 251. WEEK in REVIEW 7. Fall 2013

1. Sketch the curve $r^2 = \cos 2\theta$. Find the area inside the curve.

lemniscate



$$A = 4\iint_{D_1} dA$$

$$= 4\int_0^{\pi/4}\int_0^{\sqrt{\cos 2\theta}} r\,dr\,d\theta$$

$$= 4\int_0^{\pi/4} \frac{r^2}{2}\Big|_{r=0}^{r=\sqrt{\cos 2\theta}}\,d\theta$$

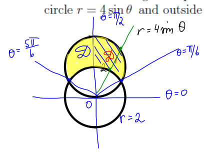
$$= 2\int_0^{\pi/4} \cos 2\theta\,d\theta$$

$$= \frac{2}{2}\sin 2\theta\Big|_0^{\pi/4} = \sin\frac{\pi}{2} = \boxed{1}$$

$x = r\cos\theta$
$y = r\sin\theta$
$dA = r\,dr\,d\theta$

$0 \le \theta \le \pi/4$
$0 \le r \le \sqrt{\cos 2\theta}$

2. Use a double integral in polar coordinates to evaluate the area of the region inside the circle $r = 4\sin\theta$ and outside the circle $r = 2$.



Points of intersection:

$4\sin\theta = 2$

$\sin\theta = \frac{2}{4} = \frac{1}{2}$

$\theta_1 = \frac{\pi}{6},\ \theta_2 = \frac{5\pi}{6}$

$\frac{\pi}{6} \le \theta \le \frac{5\pi}{6}$

For D_1:

$\frac{\pi}{6} \le \theta \le \frac{\pi}{2}$

$2 \le r \le 4\sin\theta$

$$A = \iint_D dA = 2\iint_{D_1} dA$$

$$= 2\int_{\pi/6}^{\pi/2}\int_2^{4\sin\theta} r\,dr\,d\theta$$

$$= \int_{\pi/6}^{\pi/2} r^2\Big|_{r=2}^{r=4\sin\theta}\,d\theta$$

$$= \int_{\pi/6}^{\pi/2} \left[16\sin^2\theta - 4\right]d\theta$$

$$\sin^2\theta = \frac{1-\cos 2\theta}{2}$$

$$= \int_{\pi/6}^{\pi/2} \left[8(1-\cos 2\theta) - 4\right]d\theta$$

$$= \int_{\pi/6}^{\pi/2} \left[4 - 8\cos 2\theta\right]d\theta$$

$$= \left[4\theta - 4\sin 2\theta\right]_{\theta=\pi/6}^{\theta=\pi/2}$$

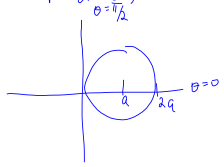
$$= 4\Big[\underbrace{\tfrac{\pi}{2} - \tfrac{\pi}{6}}_{\pi/3}\Big] - 4\Big[\underbrace{\sin\pi}_{0} - \underbrace{\sin\tfrac{\pi}{3}}_{\sqrt{3}/2}\Big]$$

$$= \boxed{4\frac{\pi}{3} + 4\frac{\sqrt{3}}{2}}$$

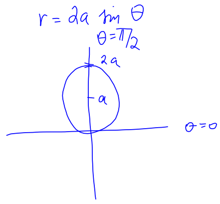
$a > 0$.

Circles:

$r = 2a\cos\theta$

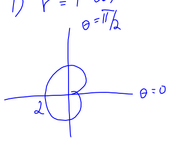


$r = 2a\sin\theta$

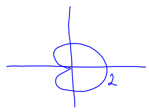


Cardioids:

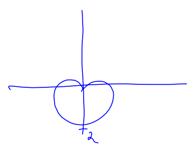
1) $r = 1 - \cos\theta$



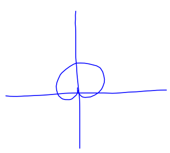
2) $r = 1 + \cos\theta$



3) $r = 1 - \sin\theta$

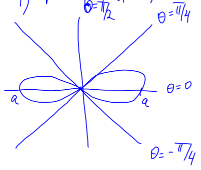


4) $r = 1 + \sin\theta$

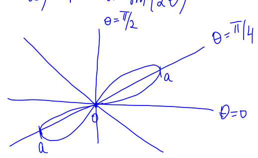


Lemniscate:

1) $r^2 = a^2\cos(2\theta)$



2) $r^2 = a^2\sin(2\theta)$



3. Use polar coordinates to evaluate

$$\int_{-1}^{1}\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \ln(x^2+y^2+1)\,dx\,dy$$

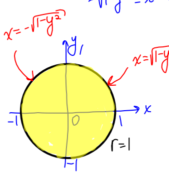
$-1 \le y \le 1$

$-\sqrt{1-y^2} \le x \le \sqrt{1-y^2}$

$x = \pm\sqrt{1-y^2}$

$x^2 = 1-y^2$

$x^2+y^2 = 1$



Polar coordinates:

$x = r\cos\theta$

$y = r\sin\theta$

$dA = r\,dr\,d\theta$

$0 \le \theta \le 2\pi$

$0 \le r \le 1$

$x^2+y^2=r^2$

$$= \frac{1}{2}\int_0^{2\pi}\left[\int_0^1 2\ln(r^2+1)\,r\,dr\right]d\theta$$

$u = r^2+1$

$du = 2r\,dr$

$r=0 \to u=1$

$r=1 \to u=2$

$$\int_a^b\int_c^d f(x)g(y)\,dy\,dx = \int_a^b f(x)\,dx \int_c^d g(y)\,dy$$

$$= \frac{1}{2}\int_0^{2\pi}\left[\int_1^2 \ln u\,du\right]d\theta$$

$$= \frac{1}{2}\underbrace{\int_0^{2\pi} d\theta}_{2\pi}\cdot\int_1^2 \ln u\,du$$

$$= \pi\int_1^2 \ln u\,du$$

integrate by parts

$\int w\,dv = wv - \int v\,dw$

$w = \ln u \qquad dv = du$

$dw = \frac{du}{u} \qquad v = u$

$$= \pi\left(\left[u\ln u\right]_1^2 - \int_1^2 u\frac{du}{u}\right)$$

$$= \pi\left(2\ln 2 - \int_1^2 du\right)$$

$$= \pi(2\ln 2 - 2 + 1)$$

$$= \boxed{\pi(2\ln 2 - 1)}$$

cylinder coord.

$$V = \int_0^{2\pi}\int_0^{4\sqrt{3}}\int_{r^2/12}^{\sqrt{64-r^2}} r\,dz\,dr\,d\theta$$

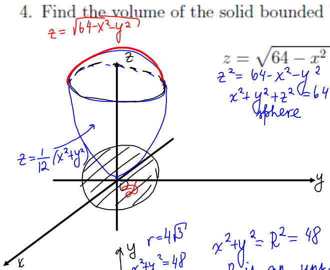
4. Find the volume of the solid bounded by the surfaces

$$z = \sqrt{64-x^2-y^2} \text{ and } z = \frac{1}{12}(x^2+y^2)$$

circular paraboloid

$z^2 = 64-x^2-y^2$

$x^2+y^2+z^2 = 64$ sphere



$$V = \iint_D \left[\sqrt{64-x^2-y^2} - \frac{1}{12}(x^2+y^2)\right]dA$$

polar coordinates:

$x = r\cos\theta$

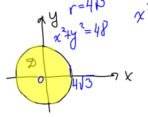
$y = r\sin\theta$

$dA = r\,dr\,d\theta$

$x^2+y^2 = r^2$

$0 \le r \le 4\sqrt{3}$

$0 \le \theta \le 2\pi$



$x^2+y^2 = R^2 = 48$

R is an unknown positive const.

$$\sqrt{64-x^2-y^2} = \frac{1}{12}(x^2+y^2)$$

$$\sqrt{64-R^2} = \frac{1}{12}R^2$$

$$64-R^2 = \frac{1}{144}R^4$$

$$R^4 + 144R^2 - 9216 = 0$$

$$R_1^2 = \frac{-144+\sqrt{144^2+4(9216)}}{2}$$

$$= \frac{-144+240}{2}$$

$$= 48$$

$$R = \sqrt{48}$$

$$= \boxed{4\sqrt{3}}$$

$$R_2^2 = \frac{-144-240}{2} < 0$$

not valid

$$V = \int_0^{2\pi}\int_0^{4\sqrt{3}}\left(\sqrt{64-r^2} - \frac{1}{12}r^2\right)r\,dr\,d\theta$$

$$= \int_0^{2\pi} d\theta \int_0^{4\sqrt{3}}\left(r\sqrt{64-r^2} - \frac{r^3}{12}\right)dr$$

$$= 2\pi\left[-\frac{1}{2}\int_0^{4\sqrt{3}} -2r\sqrt{64-r^2}\,dr - \frac{1}{12}\int_0^{4\sqrt{3}} r^3\,dr\right]$$

$u = 64-r^2$

$du = -2r\,dr$

$r=0 \to u=64$

$r=4\sqrt{3},\ u = 64-48 = 16$

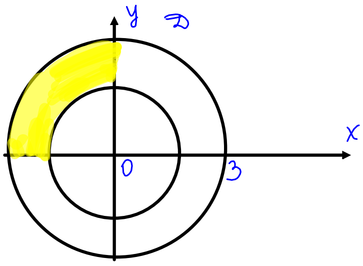
$$= -\pi\int_{64}^{16}\sqrt{u}\,du - \frac{1}{12}\left[\frac{r^4}{4}\right]_0^{4\sqrt{3}}$$

$$= \boxed{\frac{608}{3}\pi}$$

5. (a) Find the mass of the plate bounded by $x^2+y^2=4$, $x^2+y^2=9$, if $x \le 0$ and $y \ge 0$ if the density is $\rho(x,y)=\frac{y-4x}{x^2+y^2}$.

polar coordinates

$m = \iint_D \rho(x,y)\,dA$



$x = r\cos\theta$

$y = r\sin\theta$

$2 \le r \le 3$

$\frac{\pi}{2} \le \theta \le \pi$

$\rho(x,y) = \frac{r\sin - 4r\cos\theta}{r^2} = \frac{\sin\theta - 4\cos\theta}{r}$

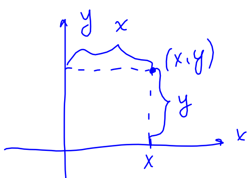
$m = \int_{\pi/2}^{\pi} \int_2^3 \frac{\sin\theta - 4\cos\theta}{\cancel{r}} \cancel{r}\,dr\,d\theta$

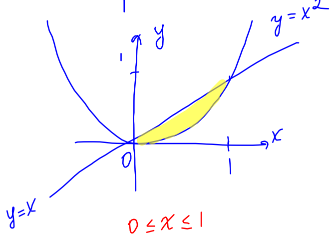
$= \dots = \boxed{5}$

(b) Find the center of mass of the plate bounded by

$$y = x^2 \text{ and } y = x$$

if the density at any point is proportional to the distance from the y-axis. $\Leftarrow \rho(x,y) = x$





$0 \le x \le 1$

$x^2 \le y \le x$

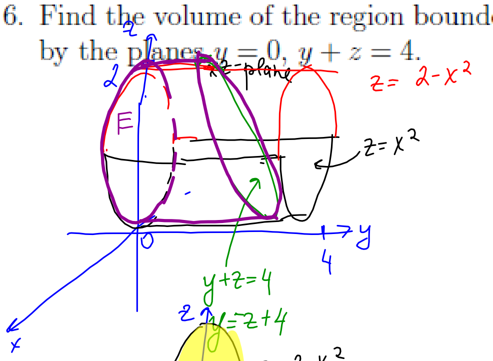
$\bar{x} = \frac{\iint_D x\rho(x,y)\,dA}{\iint_D \rho(x,y)\,dA}$, $\bar{y} = \frac{\iint_D y\rho(x,y)\,dA}{\iint_D \rho(x,y)\,dA}$

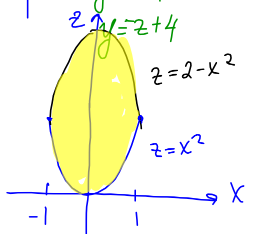
$\bar{x} = \frac{\int_0^1 \int_{x^2}^{x} x^2\,dy\,dx}{\int_0^1 \int_{x^2}^{x} x\,dy\,dx} = \dots = \frac{1/20}{1/12} = \frac{12}{20} = \boxed{\frac{3}{5}}$

$\bar{y} = \frac{\int_0^1 \int_{x^2}^{x} xy\,dy\,dx}{\int_0^1 \int_{x^2}^{x} x\,dy\,dx} = \dots = \frac{1/24}{1/12} = \boxed{\frac{1}{2}}$

$\boxed{(\bar{x},\bar{y}) = \left(\frac{3}{5}, \frac{1}{2}\right)}$

6. Find the volume of the region bounded by the parabolic cylinders $z = x^2$, $z = 2 - x^2$ and by the planes $y = 0$, $y + z = 4$.





Points of intersection:

$2-x^2 = x^2$

$x = \pm 1$

$$V = \iiint_E dV$$

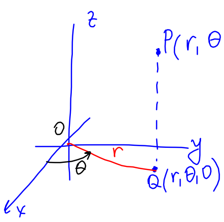
$$0 \le y \le -z+4$$
$$x^2 \le z \le 2-x^2$$
$$-1 \le x \le 1$$

$$V = \int_{-1}^{1} \int_{x^2}^{2-x^2} \int_{0}^{4-z} dy\, dz\, dx$$

$$= \dots = \boxed{8}$$

7. Write the equation $x^2 + y^2 + z^2 = 4y$ in cylindrical and spherical coordinates.

cylindrical coordinates.



z is the distance from the point to the (xz)-plane.

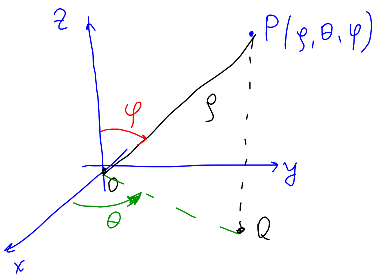
r is the distance from Q to the origin

θ is the angle between the positive x-axis and $\overrightarrow{OQ}$

$x = r\cos\theta$
$y = r\sin\theta$
$z = z$

$dV = r\, dz\, dr\, d\theta$

$x^2+y^2 = r^2$



$$\begin{cases} x = \rho\cos\theta\sin\varphi \\ y = \rho\sin\theta\sin\varphi \\ z = \rho\cos\theta \end{cases}$$

$x^2+y^2+z^2 = \rho^2$

$dV = \rho^2\sin\varphi\, d\rho\, d\varphi\, d\theta$

$x^2+y^2+z^2 = 4y$

$\boxed{r^2+z^2 = 4r\sin\theta}$

$x^2+y^2+z^2 = 4y$

$\rho^2 = 4\rho\sin\theta\sin\varphi$

$\boxed{\rho = 4\sin\theta\sin\varphi}$

8. Convert the integral

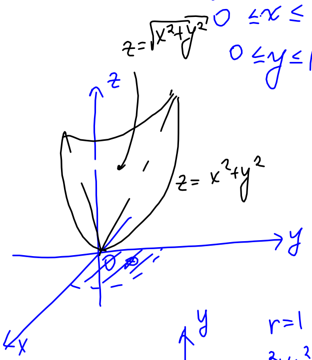
$$\int_0^1 \int_0^{\sqrt{1-y^2}} \int_{x^2+y^2}^{\sqrt{x^2+y^2}} xyz\, dz\, dx\, dy$$

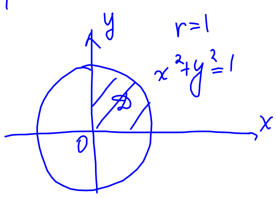
to an integral in cylindrical coordinates, but don't evaluate it.

$x^2+y^2 \le z \le \sqrt{x^2+y^2}$

$0 \le x \le \sqrt{1-y^2}$

$0 \le y \le 1$





$x = r\cos\theta$

$y = r\sin\theta$

$z = z$

$x^2+y^2 = \sqrt{x^2+y^2}$

$(x^2+y^2)^2 = x^2+y^2$

$x^2+y^2=1$

$r^2 \le z \le r$

$0 \le r \le 1$

$0 \le \theta \le \pi/2$

$$\boxed{= \int_0^{\pi/2} \int_0^1 \int_{r^2}^{r} r\cos\theta\, r\sin\theta\, z\, r\, dz\, dr\, d\theta}$$

9. Convert the integral

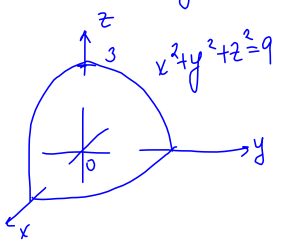
$$\int_0^3 \int_0^{\sqrt{9-y^2}} \int_0^{\sqrt{9-x^2-y^2}} (x^2+y^2+z^2)\, dz\, dx\, dy$$

to an integral in spherical coordinates, but don't evaluate it.

$0 \le z \le \sqrt{9-x^2-y^2}$

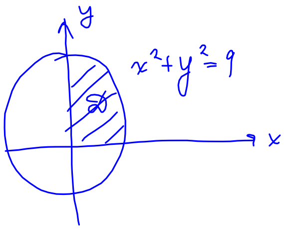
$0 \le x \le \sqrt{9-y^2}$

$0 \le y \le 3$



$z^2 = 9-x^2-y^2$

$x^2+y^2+z^2=9$



$x = \rho\cos\theta\sin\varphi$

$y = \rho\sin\theta\sin\varphi$

$z = \rho\cos\varphi$

$dz\,dx\,dy = \rho^2\sin\varphi\, d\rho\, d\varphi\, d\theta$

$x^2+y^2+z^2 = \rho^2$

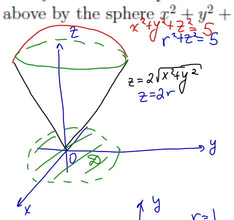
$0 \le \theta \le \pi/2$

$0 \le \rho \le 3$

$0 \le \varphi \le \pi/2$

$$\boxed{= \int_0^{\pi/2} \int_0^{\pi/2} \int_0^3 \rho\, \rho^2 \sin\varphi\, d\rho\, d\varphi\, d\theta}$$

10. Use cylindrical and spherical coordinates to find the volume of the ice-cream cone bounded above by the sphere $x^2 + y^2 + z^2 = 5$ and below by the cone $z = 2\sqrt{x^2 + y^2}$.

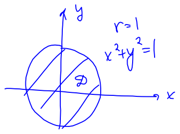


$z^2 = 5 - x^2 - y^2$

$z^2 = 4(x^2+y^2)$

$5 - x^2 - y^2 = 4(x^2+y^2)$

$x^2+y^2=1$



cylindrical coordinates:

$x = r\cos\theta$

$y = r\sin\theta$

$z = z$

$z = 2\sqrt{x^2+y^2} = 2r$ cone

$z = \sqrt{5-r^2}$ – sphere

$2r \le z \le \sqrt{5-r^2}$

$0 \le r \le 1$

$0 \le \theta \le 2\pi$

$$V = \int_0^{2\pi}\int_0^1\int_{2r}^{\sqrt{5-r^2}} r\,dz\,dr\,d\theta = \ldots = \boxed{\frac{2\pi}{3}(5\sqrt{5}-10)}$$

spherical coordinates:

$$\begin{cases} x = \rho\cos\theta\sin\varphi \\ y = \rho\sin\theta\sin\varphi \\ z = \rho\cos\varphi \end{cases}$$

$x^2+y^2+z^2=5$

$\rho^2 = 5$

$\boxed{\rho = \sqrt{5}}$

$z = 2\sqrt{x^2+y^2}$

$\rho\cos\varphi = 4\rho^2\sin^2\varphi$

$\cos^2\varphi = 4\sin^2\varphi$

$\boxed{\tan\varphi = \frac{1}{2}}$

$\varphi = \tan^{-1}\left(\frac{1}{2}\right)$

$0 \le \rho \le \sqrt{5}$

$0 \le \varphi \le \tan^{-1}\left(\frac{1}{2}\right)$

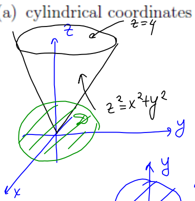
$0 \le \theta \le 2\pi$

$$V = \int_0^{2\pi}\int_0^{\tan^{-1}(\frac{1}{2})}\int_0^{\sqrt{5}} \rho^2\sin\varphi\,d\rho\,d\varphi\,d\theta = \boxed{\frac{2\pi}{3}[5\sqrt{5}-10]}$$

$$= \int_0^{2\pi} d\theta \cdot \int_0^{\tan^{-1}(\frac{1}{2})} \sin\varphi\,d\varphi \cdot \int_0^{\sqrt{5}} \rho^2\,d\rho$$

11. Let E be the solid region bounded below by the cone $z^2 = x^2 + y^2$, $z \ge 0$ and above by the plane $z = 4$. Set up the integral for computing the mass of the solid if the density is $\rho(x, y, z) = xyz$ in

(a) cylindrical coordinates



$x^2+y^2=r^2$

$x = r\cos\theta$

$y = r\sin\theta$

$z = z.$

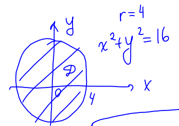
intersection:

$x^2+y^2=16$

$\sqrt{x^2+y^2} \le z \le 4$

$0 \le r \le 4$

$0 \le \theta \le 2\pi$



$$\boxed{m = \iiint_E \rho(x,y,z)\,dV = \int_0^{2\pi}\int_0^4\int_r^4 r\cos\theta\, r\sin\theta\, z\, r\,dz\,dr\,d\theta}$$

(b) in spherical coordinates

$x = \rho\cos\theta\sin\varphi$

$y = \rho\sin\theta\sin\varphi$

$z = \rho\cos\varphi$

$z^2 = x^2+y^2$

$\rho^2\cos^2\varphi = \rho^2\sin^2\varphi$

$\cos^2\varphi = \sin^2\varphi$

$\tan^2\varphi = 1$

$\boxed{\varphi = \pi/4}$ cone

$z = 4$

$\rho\cos\varphi = 4$

$\boxed{\rho = \frac{4}{\cos\varphi}}$ plane

$0 \le \rho \le \frac{4}{\cos\varphi}$

$0 \le \varphi \le \pi/4$

$0 \le \theta \le 2\pi$

$$\boxed{V = \int_0^{2\pi}\int_0^{\pi/4}\int_0^{\frac{4}{\cos\varphi}} \rho\cos\theta\sin\varphi\,\rho\sin\theta\sin\varphi\,\rho\cos\varphi\,\rho^2\sin\varphi\,d\rho\,d\varphi\,d\theta}$$

12. Find the center of mass of a solid right circular cone (the centroid of the cone) with height H and base radius R assuming that the density is homogeneous.

pospone to WIR #8.