Math 251. WEEK in REVIEW 8. Fall 2013

1. Find the gradient vector field of $f(x, y, z) = x \ln(y^4 + z^2)$

$\nabla f(x,y,z) = \langle f_x, f_y, f_z \rangle$

$f_x = \ln(y^4+z^2)$

$f_y = \frac{x}{y^4+z^2} \cdot \frac{\partial}{\partial y}(y^4)$

$= \frac{4y^3 x}{y^4+z^2}$

$f_z = \frac{x}{y^4+z^2} \cdot \frac{\partial}{\partial z}(z^2)$

$= \frac{2zx}{y^4+z^2}$

$$\boxed{\nabla f(x,y,z) = \left\langle \ln(y^4+z^2), \frac{4y^3x}{y^4+z^2}, \frac{2zx}{y^4+z^2} \right\rangle}$$

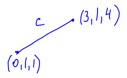
2. Let C be the line segment starting at $(0, 1, 1)$ and ending at $(3, 1, 4)$. Find the mass of a thin wire in the shape of C with the density $\rho(x, y) = x + y$.

$m = \int_C \rho(x,y)\,ds$

$C: x = x(t), y = y(t), z = z(t), a \le t \le b$

$\int_C f(x,y,z)\,ds = \int_a^b f(x(t),y(t),z(t))\sqrt{[x'(t)]^2+[y'(t)]^2+[z'(t)]^2}\,dt$

$m = \int_C (x+y)\,ds$



Parametric equations for C.

$C \parallel \vec{v} = \langle 3,0,3 \rangle$

$\vec{r}(t) = \langle x,y,z \rangle = \langle 0,1,1 \rangle + t\langle 3,0,3 \rangle$

$\vec{r}(0) = \langle 0,1,1 \rangle$

$\vec{r}(1) = \langle 3,1,4 \rangle$

$\begin{cases} x = 3t \\ y = 1 \\ z = 1+3t \end{cases}$ $0 \le t \le 1$

$x'(t) = 3$
$y'(t) = 0$
$z'(t) = 3.$

$m = \int_C (x+y)\,ds = \int_a^b (x(t)+y(t))\sqrt{[x'(t)]^2+[y'(t)]^2+[z'(t)]^2}\,dt$

$= \int_0^1 (3t+1)\sqrt{9+9}\,dt$

$= \sqrt{18}\int_0^1 (3t+1)\,dt$

$= 3\sqrt{2}\left(\frac{3t^2}{2}+t\right)\Big]_0^1$

$= \boxed{3\sqrt{2}\left(\frac{3}{2}+1\right)}$

3. Find the mass of a thin wire in the shape of C with the density $\rho(x, y, z) = 7y^2z$ if C is given by $\vec{r}(t) = \langle \frac{2}{3}t^3, t, t^2 \rangle$, $0 \le t \le 1$.

$m = \int_C \rho(x,y,z)\,ds = \int_0^1 \rho(\overset{7y^2z}{x(t), y(t), z(t)})\sqrt{[x'(t)]^2+[y'(t)]^2+[z'(t)]^2}\,dt$

$x(t) = \frac{2}{3}t^3$	$x'(t) = 2t^2$
$y(t) = t$	$y'(t) = 1$
$z(t) = t^2$	$z'(t) = 2t$

$= \int_0^1 7(\overset{y^2}{t^2})\overset{z}{t^2}\underbrace{\sqrt{4t^4+1+4t^2}}_{(2t^2+1)^2}\,dt$

$= \int_0^1 7t^4(2t^2+1)\,dt$

$= \int_0^1 (14t^6+7t^4)\,dt$

$= \left[\frac{14t^7}{7} + \frac{7t^5}{5}\right]_0^1$

$= 2 + \frac{7}{5}$

$= \boxed{\frac{17}{5}}$

4. Find the line integral of the vector field $\vec{F}(x, y, z) = \langle -yz^2, xz^2, z^3 \rangle$ around the circle $\vec{r}(t) = \langle 2\cos t, 2\sin t, 8 \rangle$.

scalar product

$C: \vec{r}(t)$, $a \le t \le b$ $\quad \int_C \vec{F} \cdot d\vec{r} = \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\,dt$

$x(t) = 2\cos t$
$y(t) = 2\sin t$
$z(t) = 8$

$0 \le t \le 2\pi$

$\vec{r}(t) = \langle 2\cos t, 2\sin t, 8 \rangle$
$\vec{r}\,'(t) = \langle -2\sin t, 2\cos t, 0 \rangle$

$\vec{F}(x,y,z) = \langle -yz^2, xz^2, z^3 \rangle$

$= \langle -2\sin t\,(8^2), 2\cos t\,(8^2), 8^3 \rangle$

$= \langle -128\sin t, 128\cos t, 512 \rangle = \vec{F}(\vec{r}(t))$

$\vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t) = \langle -128\sin t, 128\cos t, 512 \rangle \cdot \langle -2\sin t, 2\cos t, 0 \rangle$

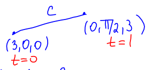
$= 256\sin^2 t + 256\cos^2 t + 0$

$= \boxed{256}$

$\int_C \vec{F} \cdot d\vec{r} = \int_0^{2\pi} 256\,dt = \boxed{512\pi}$

5. Find the work done by the force field $\vec{F}(x,y,z) =< z, x, y >$ in moving a particle from the point $(3, 0, 0)$ to the point $(0, \pi/2, 3)$

(a) along the straight line



$W = \int_C \vec{F} \cdot d\vec{r}$

Parametric equations of the line:

$C \parallel \vec{v} = < -3, \pi/2, 3 >$

$\vec{r}(t) = < x, y, z > = < 3, 0, 0 > + t < -3, \pi/2, 3 >$

$= < 3 - 3t, \pi/2 \, t, 3t >$

$\vec{r}\,'(t) = < -3, \pi/2, 3 >$

$0 \le t \le 1$

$x(t) = 3 - 3t$

$y(t) = \frac{\pi}{2} t$

$z(t) = 3t$

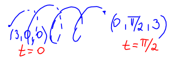
$\vec{F}(x,y,z) = < z, x, y > = < 3t, 3-3t, \frac{\pi}{2} t >$

$\vec{F} \cdot \vec{r}\,' = < 3t, 3-3t, \frac{\pi}{2} t > \cdot < -3, \frac{\pi}{2}, 3 > = -9t + \frac{3\pi}{2} - \frac{3\pi}{2} t + \frac{3\pi}{2} t$

$= \frac{3\pi}{2} - 9t$

$W = \int_C \vec{F} \cdot d\vec{r} = \int_0^1 \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\, dt = \int_0^1 \left(\frac{3\pi}{2} - 9t\right) dt = \left[\frac{3\pi}{2} t - \frac{9t^2}{2}\right]_0^1 = \boxed{\frac{3\pi}{2} - \frac{9}{2}}$

(b) along the helix $x = 3\cos t$, $y = t$, $z = 3\sin t$



$\vec{r}(t) = < 3\cos t, t, 3\sin t >$

$0 \le t \le \pi/2$

$\vec{r}\,'(t) = < -3\sin t, 1, 3\cos t >$

$W = \int_C \vec{F} \cdot d\vec{r} = \int_0^{\pi/2} F(\vec{r}(t)) \cdot \vec{r}\,'(t)\, dt = \int_0^{\pi/2} < 3\sin t, 3\cos t, t > \cdot < -3\sin t, 1, 3\cos t > dt$

$\sin^2 t = \frac{1 - \cos 2t}{2}$

$= \int_0^{\pi/2} \left[-9\sin^2 t + 3\cos t + 3t \cos t\right] dt$

$= \int_0^{\pi/2} \left[-\frac{9}{2}(1 - \cos 2t) + 3\cos t\right] dt + 3\int_0^{\pi/2} t \cos t\, dt$

integrate by parts.

$u = t \quad du = dt$

$dv = \cos t\, dt \quad v = \sin t$

$\int_a^b u\,dv = uv\Big]_a^b - \int_a^b v\,du$

$= \left(-\frac{9}{2}\left[t - \frac{1}{2}\sin 2t\right] + 3\sin t\right)\Big]_0^{\pi/2} + 3\left[t \sin t\Big]_0^{\pi/2} - \int_0^{\pi/2} \sin t\, dt\right]$

$= -\frac{9}{2}\left(\frac{\pi}{2}\right) + 3 \quad + 3\left(\frac{\pi}{2} + \cos t\Big]_0^{\pi/2}\right)$

$= -\frac{9}{4}\pi + 3 + \frac{3\pi}{2} - 3$

$= -\frac{9\pi}{4} + \frac{3\pi}{2}$

$= \boxed{-\frac{3\pi}{4}}$

6. Find a scalar function $f(x, y, z)$ such that $\nabla f =< \overset{f_x}{2xy + z}, \overset{f_y}{x^2 - 2y}, \overset{f_z}{x} >$ and $f(1, 2, 0) = 3$.

$f_x = 2xy + z$

$f_y = x^2 - 2y$

$f_z = x$

$f_z = x$

$f = \int x\, dz$

$f = xz + g(x,y)$

$f_x = \cancel{z} + g_x = 2xy + \cancel{z}$

$g_x = 2xy$

$g = \int 2xy\, dx$

$= x^2 y + h(y)$

update f:

$f(x,y,z) = xz + x^2 y + h(y)$

$f_y = \cancel{x^2} + h'(y) = \cancel{x^2} - 2y$

$h'(y) = -2y$

$h(y) = \int (-2y)\, dy$

$= -y^2 + C$

update f:

$f(x,y,z) = xz + x^2 y - y^2 + C$

Use $f(1,2,0) = 3$.

$f(1,2,0) = 0 + 2 - 4 + C = 3$

$\boxed{C = 5}$

$\boxed{f(x,y,z) = xz + x^2 y - y^2 + 5}$

7. Let $\vec{F}(x,y) = \langle \overset{P(x,y)}{x^3y^4}, \overset{Q(x,y)}{x^4y^3} \rangle$. Compute $\int_C \vec{F} \cdot d\vec{r}$ where

(a) C is any simple closed path.

$\oint_C \vec{F} \cdot d\vec{r}$ Green's Theorem

$$= \iint_{\mathcal{D}} \left[\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right] dA$$

$P(x,y) = x^3y^4$

$Q(x,y) = x^4y^3$

$\vec{F}$ is a conservative vector field.

$$\frac{\partial Q}{\partial x} = 4x^3y^3 = \frac{\partial P}{\partial y} = 4x^3y^3$$

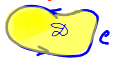
$$\iint_{\mathcal{D}} [4x^3y^3 - 4x^3y^3]\, dA = \boxed{0}$$

Green's Theorem.

Let C be positively oriented smooth simple closed curve given by $\vec{r}(t)$, $a \le t \le b$.

$\vec{r}(a) = \vec{r}(b)$.

and let $\mathcal{D}$ is the region boulded by C

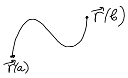


Then

$$\underbrace{\oint_C \vec{F} \cdot d\vec{r}}_{\text{line integral.}} = \underbrace{\iint_{\mathcal{D}} \left[\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right] dA}_{\text{double integral.}}$$

(b) C is any path from the point $M(0,0)$ to the point $N(1,2)$.

Fundamental theorem for line integrals.



$C: \vec{r}(t)$, $a \le t \le b$.

$$\int_C \nabla f \cdot d\vec{r} = f(\vec{r}(b)) - f(\vec{r}(a))$$

$F(x,y) = \langle x^3y^4, x^4y^3 \rangle$.

Find the potential function for $F(x,y)$.

Find a function f such that $\nabla f = F(x,y) = \langle x^3y^4, x^4y^3 \rangle$

$\nabla f = \langle f_x, f_y \rangle$

$$\begin{cases} f_x = x^3y^4 \\ f_y = x^4y^3 \end{cases}$$

$f_x = x^3y^4$

$f = \int x^3y^4\, dx$

$= \frac{x^4}{4} y^4 + g(y)$

$f(x,y) = \frac{1}{4}x^4y^4 + g(y)$

$f_y = x^4y^3 + g'(y) = x^4y^3$

$g'(y) = 0$

$g(y) = C$

$f(x,y) = \frac{1}{4}x^4y^4 + \cancel{C}^{\,0}$

$\int_C \vec{F} \cdot d\vec{r} = f(N) - f(M)$

$= f(1,2) - f(0,0)$

$= \frac{1 \cdot 2^4}{4} = \boxed{4}$

8. Let $\vec{F}(x,y) = < x + y^2, 2xy + y^2 >$.

(a) Show that $\vec{F}$ is conservative vector field.

$\vec{F} = \langle P(x,y), Q(x,y) \rangle$ is conservative if and only if $\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$

$P(x,y) = x + y^2$ $\qquad \frac{\partial P}{\partial y} = 2y$

$Q(x,y) = 2xy + y^2$ $\qquad \frac{\partial Q}{\partial x} = 2y$

$\boxed{\text{conservative}}$

(b) Compute $\int_C \vec{F} \cdot d\vec{r}$ where C is any path from (-1,0) to (2,2).

Find function $f(x,y)$ such that $\nabla f = \langle x + y^2, 2xy + y^2 \rangle$.

$$\begin{cases} f_x = x + y^2 \\ f_y = 2xy + y^2 \end{cases}$$

$f = \int (x + y^2)\, dx$

$= \frac{x^2}{2} + y^2x + g(y)$

$f_y = 2xy + g'(y) = 2xy + y^2$

$g'(y) = y^2$

$g(y) = \frac{y^3}{3} + C$

$f(x,y) = \frac{x^2}{2} + y^2x + \frac{y^3}{3} + \cancel{C}^{\,0}$

$\int_C \vec{F} \cdot d\vec{r} = f(2,2) - f(-1,0)$

$= \frac{4}{2} + 4(2) + \frac{8}{3} - \frac{1}{2}$

$= \boxed{\frac{73}{6}}$

9. Let $\vec{F}(x,y) = < \overset{P}{2x + y^2 + 3x^2y}, \overset{Q}{2xy + x^3 + 3y^2} >$.

(a) Show that $\vec{F}$ is conservative vector field.

$$\frac{\partial P}{\partial y} = 2y + 3x^2$$
$$\|$$
$$\frac{\partial Q}{\partial x} = 2y + 3x^2$$

$\boxed{\text{conservative}}$

(b) Evaluate $\int_C \vec{F} \cdot d\vec{r}$ where C is the arc of the curve $\boxed{y = x \sin x}$ from $(0,0)$ to $(\pi, 0)$.

Integrals of conservative vector fields are independent of path of integration.

$$\int_C \vec{F} \cdot d\vec{r} = f(\pi,0) - f(0,0), \text{ where } \nabla f = \vec{F}.$$

$$\nabla f = < 2x + y^2 + 3x^2y, 2xy + x^3 + 3y^2 >$$

$$\begin{cases} f_x = 2x + y^2 + 3x^2 y \\ f_y = 2xy + x^3 + 3y^2 \end{cases}$$

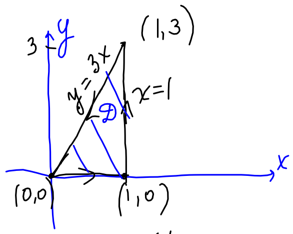
$$f(x,y) = \int (2x + y^2 + 3x^2 y)\,dx$$
$$= x^2 + y^2 x + x^3 y + g(y)$$

$$f_y = 2xy + x^3 + g'(y) = 2xy + x^3 + 3y^2$$
$$g'(y) = 3y^2$$
$$g(y) = y^3 + C$$

$$f(x,y) = x^2 + y^2 x + x^3 y + y^3 + \cancel{C}^{\,0}$$

$$\int_C \vec{F} \cdot d\vec{r} = f(\pi, 0) - f(0,0) = \boxed{\pi^2}$$

10. Compute the integral $I = \oint_C (\cos(x^4) + xy)\,dx + (y^2 e^y + x^2)\,dy$, where C is the triangular curve consisting of the line segments from $(0,0)$ to $(1,0)$, from $(1,0)$ to $(1,3)$, and from $(1,3)$ to $(0,0)$.



$0 \le x \le 1$
$0 \le y \le 3x$

Green's theorem.

$$\oint_C P\,dx + Q\,dy = \iint_D \left[\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right] dA, \quad \partial D = C.$$

$P(x,y) = \cos(x^4) + xy$; $\frac{\partial P}{\partial y} = x$

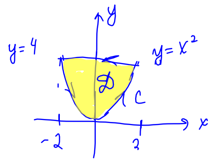
$Q(x,y) = y^2 e^y + x^2$; $\frac{\partial Q}{\partial x} = 2x$

$$I = \iint_D [2x - x]\,dA$$
$$= \int_0^1 \int_0^{3x} x\,dy\,dx$$
$$= \int_0^1 xy \Big]_{y=0}^{y=3x} dx$$
$$= \int_0^1 3x^2\,dx = x^3 \Big]_0^1 = \boxed{1}$$

11. Compute the integral along the given positively oriented curve C:

$$\int_C (y^2 - \arctan x)\, dx + (3x + \sin y)\, dy,$$

where C is the boundary of the region enclosed by the parabola $y = x^2$ and the line $y = 4$.



$-2 \le x \le 2$

$x^2 \le y \le 4$

C is closed.

Green's theorem:

$$\int_C P\,dx + Q\,dy = \iint_D \left[\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right] dA$$

$P(x,y) = y^2 - \arctan x, \quad \frac{\partial P}{\partial y} = 2y$

$Q(x,y) = 3x + \sin y, \quad \frac{\partial Q}{\partial x} = 3$

$$\int_C (y^2 - \arctan x)\,dx + (3x + \sin y)\,dy$$

$$= \int_{-2}^{2}\int_{x^2}^{4} (3 - 2y)\,dy\,dx$$

$$= \int_{-2}^{2} \left[3y - y^2\right]_{y=x^2}^{y=4} dx$$

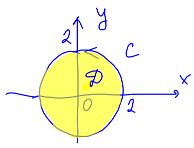
$$= \int_{-2}^{2} \left[-4 - 3x^2 + x^4\right] dx$$

$$= \left[-4x - \frac{3x^3}{3} + \frac{x^5}{5}\right]_{-5}^{5} = \boxed{\frac{64}{5} - 32}$$

12. Compute the integral

$$\int_C (\overset{P}{12 - x^2 y - y^3 + \tan x})\, dx + (\overset{Q}{xy^2 + x^3 - e^y})\, dy$$

where C is positively oriented boundary of the region enclosed by the circle $x^2 + y^2 = 4$. Sketch the curve C indicating the positive direction.



Green's Theorem.

$\frac{\partial Q}{\partial x} = y^2 + 3x^2$

$\frac{\partial P}{\partial y} = -x^2 - 3y^2$

$$\int_C (12 - x^2 y - y^3 + \tan x)\,dx + (xy^2 + x^3 - e^y)\,dy$$

$$= \iint_D \left[y^2 + 3x^2 + x^2 + 3y^2\right] dA$$

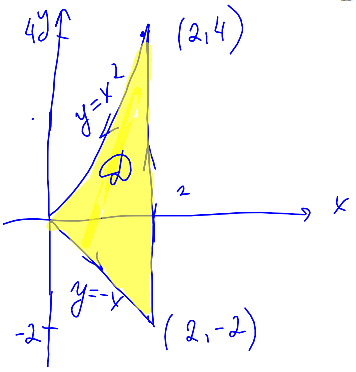
$$= 4\iint_D \left[y^2 + x^2\right] dA$$

Polar coord: $x = r\cos\theta$, $y = r\sin\theta$, $dA = r\,dr\,d\theta$, $x^2 + y^2 = r^2$, $\quad 0 \le r \le 2,\ 0 \le \theta \le 2\pi$

$$= 4\int_0^{2\pi}\int_0^{2} r(r^2)\,dr\,d\theta$$

$$= \boxed{4(2\pi)\frac{8}{3}}$$

13. Given the line integral $I = \int_C 4x^2y\,dx - (2+x)\,dy$ where C consists of the line segment from $(0,0)$ to $(2,-2)$, the line segment from $(2,-2)$ to $(2,4)$, and the part of the parabola $y = x^2$ from $(2,4)$ to $(0,0)$. Use Green's theorem to **evaluate** the given integral and **sketch** the curve C indicating the *positive direction*.



$0 \le x \le 2$
$-x \le y \le x^2$

$P = 4x^2y \qquad \frac{\partial P}{\partial y} = 4x^2$

$Q = -(2+x) \qquad \frac{\partial Q}{\partial x} = -1$

$$I = \int_C 4x^2y\,dx - (2+x)\,dy$$

$$= \iint_D (-1 - 4x^2)\,dA$$

$$= \int_0^2 \int_{-x}^{x^2} (-1 - 4x^2)\,dy\,dx$$

$$= \int_0^2 (-1 - 4x^2)(x^2 + x)\,dx$$

$$= \ldots$$