MATH 251 W.I.R. 9 FALL 2013

1. Given the vector field $\vec{F} = z\vec{\imath} + 2yz\vec{\jmath} + (x+y^2)\vec{k}$.

(a) Find the divergence of the field.

$\nabla = \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \rangle$, $\vec{F} = \langle P(x,y,z), Q(x,y,z), R(x,y,z) \rangle$

$$\boxed{div\,\vec{F} = \nabla \cdot \vec{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}}$$

$\vec{F} = \langle z, 2yz, x+y^2 \rangle$

$$div\,\vec{F} = \frac{\partial (z)}{\partial x} + \frac{\partial}{\partial y}(2yz) + \frac{\partial}{\partial z}(x+y^2) = 0 + 2z + 0 = \boxed{2z}$$

(b) Find the curl of the field.

$$curl\,\vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} = \boxed{\vec{i}\left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right) - \vec{j}\left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right) + \vec{k}\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)}$$

$$curl\,\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z & 2yz & x+y^2 \end{vmatrix} = \vec{i}\left(\frac{\partial}{\partial y}(x+y^2) - \frac{\partial}{\partial z}(2yz)\right) - \vec{j}\left(\frac{\partial}{\partial x}(x+y^2) - \frac{\partial}{\partial z}(z)\right)$$
$$+ \vec{k}\left(\frac{\partial}{\partial x}(2yz) - \frac{\partial}{\partial y}(z)\right) = \vec{i}(2y-2y) - \vec{j}(1-1) + \vec{k}(0-0)$$
$$= \boxed{\langle 0,0,0 \rangle}$$

(c) Is the given field conservative? If it is, find a potential function.

Since $curl\,\vec{F} = \langle 0,0,0 \rangle$, then $\vec{F}$ is conservative.

Find f such that $\nabla f = \vec{F}$

$\langle f_x, f_y, f_z \rangle = \langle z, 2yz, x+y^2 \rangle$

$$\begin{cases} f_x = z \\ f_y = 2yz \\ f_z = x+y^2 \end{cases}$$

$\int f_x dx = \int z\,dx$

$f(x,y,z) = xz + g(y,z)$, g is unknown

Plug f into the 2nd equation.

Find $\frac{\partial f}{\partial y}$: $f_y = 0 + g_y$

$\int g_y dy = \int 2yz\,dy$

$g(y,z) = y^2 z + h(z)$, h is unknown

update f: $f(x,y,z) = xz + y^2 z + h(z)$

Plug f into the 3rd equation.

$f_z = x + y^2 + h'(z)$

$\cancel{x} + \cancel{y^2} + h'(z) = \cancel{x} + \cancel{y^2}$

$h'(z) = C$ (C is an arbitrary const.)

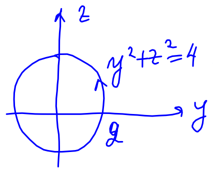
$$\boxed{f(x,y,z) = xz + y^2 z + \cancel{C}^{\,0}}$$

if $\nabla f = \vec{F}$, then $\oint_C \vec{F} \cdot d\vec{r} = 0$

(d) Compute $\int_C z\,\cancel{dx}^{\,0} + 2yz\,dy + (\cancel{x}^{\,5} + y^2)\,dz$ where C is the positively oriented curve $y^2 + z^2 = 4$, $x = 5$.

$dx = 0$

$\int_C 2yz\,dy + (5+y^2)\,dz = 0$ by the Fundamental Theorem.



$y^2+z^2=4$ (closed curve)

(e) Compute $\int_C z\,dx + 2yz\,dy + (x+y^2)\,dz$ where C consists of the three line segments: from $(0,0,0)$ to $(4,0,0)$, from $(4,0,0)$ to $(2,3,1)$, and from $(2,3,1)$ to $(1,1,1)$.

initial point ... terminal point.

$$\int_C z\,dx + 2yz\,dy + (x+y^2)\,dz = f(1,1,1) - f(0,0,0)$$

where $f(x,y,z) = xz + y^2 z$ (part (c))

$= 1 + 1 - 0$

$= \boxed{2}$

2. Is there a vector field such that curl$F = -2x\vec{i} + 3yz\vec{j} - xz^2 k$?

div (curl $\vec{F}$) = 0 for any vector field $\vec{F}$.

$$\text{div}(\text{curl } \vec{F}) = \frac{\partial}{\partial x}(-2x) + \frac{\partial}{\partial y}(3yz) + \frac{\partial}{\partial z}(-xz^2)$$

$$= -2 + 3z - 2xz \neq 0$$

$\boxed{NO}$

3. Identify the surface $x = \sqrt{u}\cos v$, $y = 5u$, $z = \sqrt{u}\sin v$. Eliminate u and v.

$$x^2 + z^2 = u\cos^2 v + u\sin^2 v$$

$$x^2 + z^2 = u$$

$$u = \frac{y}{5}$$

$$\boxed{x^2 + z^2 = \frac{y}{5}}$$

circular paraboloid oriented along the y-axis.

4. Identify the surface which is the graph of the vector-function $\vec{r}(u,v) =< u + v, u - v, u >$.

x y z

$x = u + v$

$y = u - v$

$z = u$

Eliminate u and v.

$$\begin{array}{r} x = u + v \\ + \quad y = u - v \\ \hline x + y = 2u \end{array}$$

$\boxed{x + y = 2z}$ plane

5. Find a parametric representation of the following surfaces:

(a) $x + 2y + 3z = 0$;

$$\begin{cases} x = -2y - 3z \\ y = y \\ z = z \end{cases}$$

y and z are parameters

(b) the portion of the plane $x + 2y + 3z = 0$ in the first octant;

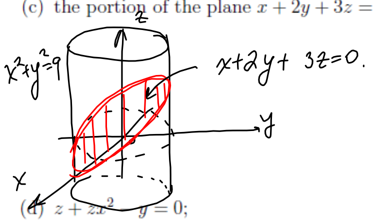
$$\begin{cases} x = -2y - 3z \\ y = y \\ z = z \end{cases}$$

$y \geq 0$, $x \geq 0$
$z \geq 0$;

$-2y - 3z \geq 0$
$y \leq -\frac{3z}{2}$

Parameter domain:
$\mathcal{D} = \{(y, z) \mid z \geq 0,\ y \leq -\frac{3z}{2}\}$

(c) the portion of the plane $x + 2y + 3z = 0$ inside the cylinder $x^2 + y^2 = 9$;



The parameter domain is $x^2 + y^2 \leq 9$

$$\begin{cases} x = x \\ y = y \\ z = -\frac{1}{3}(x + 2y) \end{cases}$$

where $x^2 + y^2 \leq 9$

(d) $z + 2x^2 - y = 0$;

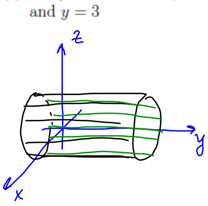
(d) $z + 2x^2 - y = 0$

$$\begin{cases} x = x \\ y = z + 2x^2 \\ z = z \end{cases}$$

(e) $y = x^2$;

$$\begin{cases} x = x \\ y = x^2 \\ z = z \end{cases}$$

(f) the portion of the cylinder $x^2 + z^2 = 25$ that extends between the planes $y = -1$ and $y = 3$



cylindrical coord.

$$\begin{cases} x = r\cos\theta \\ y = y \\ z = r\sin\theta \end{cases}$$

$x^2 + z^2 = r^2 = 25$
$r = 5$

$$\begin{cases} x = 5\cos\theta \\ y = y \\ z = 5\sin\theta \end{cases}$$

$-1 \leq y \leq 3$,

6. Find an equation of the plane tangent to the surface $x = u, y = 2v, z = u^2 + v^2$ at the point $(1, 4, 5)$.

If the surface is given by $\vec{r}(u,v)$, then the normal vector to the surface $\vec{r}(u,v)$ at the point (u_0, v_0) is

$$N(u_0,v_0) = \vec{r}_u(u_0,v_0) \times \vec{r}_v(u_0,v_0)$$

normal vector is $\perp$ to the tangent plane to the surface at (u_0, v_0)

$\vec{r}(u,v) = \langle u, 2v, u^2+v^2 \rangle$

① Find u_0 and v_0

$\vec{r}(u_0,v_0) = \langle 1,4,5 \rangle$

$\langle u_0, 2v_0, u_0^2+v_0^2 \rangle = \langle 1,4,5 \rangle$

$\boxed{u_0 = 1}$

$2v_0 = 4$ $\quad \boxed{v_0 = 2}$

② Find the normal vector $\vec{N}(1,2) = \vec{r}_u(1,2) \times \vec{r}_v(1,2)$

$\vec{r}(u,v) = \langle u, 2v, u^2+v^2 \rangle$

$\vec{r}_u(u,v) = \langle 1, 0, 2u \rangle$ $\qquad \vec{r}_u(1,2) = \langle 1, 0, 2 \rangle$

$\vec{r}_v(u,v) = \langle 0, 2, 2v \rangle$ $\qquad \vec{r}_v(1,2) = \langle 0, 2, 4 \rangle$

$$\vec{N}(1,2) = \langle 1,0,2 \rangle \times \langle 0,2,4 \rangle$$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & 2 \\ 0 & 2 & 4 \end{vmatrix}$$

$$= -4\vec{i} - 4\vec{j} + 2\vec{k} = \langle -4, -4, 2 \rangle$$

③ Write an equation of the tangent plane

$$\boxed{-4(x-1) - 4(y-4) + 2(z-5) = 0}$$

$S: x^2+y^2+z^2=4$

7. Given a sphere of radius 2 centered at the origin, find an equation for the plane tangent to it at the point $(1, 1, \sqrt{2})$ considering the sphere as

(a) the graph of $g(x, y) = \sqrt{4 - x^2 - y^2}$ (Note that the graph of $g(x, y)$ is the upper half-sphere);

$z = g(x,y)$, $\vec{N} = \langle g_x, g_y, -1 \rangle$

$$= \left\langle -\frac{x}{\sqrt{4-x^2-y^2}}, -\frac{y}{\sqrt{4-x^2-y^2}}, -1 \right\rangle$$

$$\vec{N}(1,1) = \left\langle -\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, -1 \right\rangle$$

an equation of the tangent plane:

$$\boxed{-\frac{1}{\sqrt{2}}(x-1) - \frac{1}{\sqrt{2}}(y-1) - (z-\sqrt{2}) = 0}$$

(b) a level surface of $f(x, y, z) = x^2 + y^2 + z^2$;

$$N(x,y,z) = \langle f_x, f_y, f_z \rangle = \nabla f(x,y,z)$$

$$= \langle 2x, 2y, 2z \rangle$$

$$\vec{N}(1,1,\sqrt{2}) = \langle 2, 2, 2\sqrt{2} \rangle$$

An equation of the tangent plane.

$$\boxed{2(x-1) + 2(y-1) + 2\sqrt{2}(z-\sqrt{2}) = 0}$$

(c) a surface parametrized by the spherical coordinates.

$$x^2+y^2+z^2=4$$

$$\begin{cases} x=\rho\cos\theta\sin\varphi \\ y=\rho\sin\theta\sin\varphi \\ z=\rho\cos\varphi \end{cases}$$

$$x^2+y^2+z^2=\rho^2$$

$$\rho^2=4$$
$$\rho=2$$

Parametric equations for the sphere

$$\begin{cases} x=2\cos\theta\sin\varphi \\ y=2\sin\theta\sin\varphi \\ z=2\cos\varphi \end{cases}$$

$$\vec{r}(\theta,\varphi)=\langle 2\cos\theta\sin\varphi,\ 2\sin\theta\sin\varphi,\ 2\cos\varphi\rangle$$

① Find θ_0 and φ_0 such that

$$\vec{r}(\theta_0,\varphi_0)=(1,1,\sqrt{2})=\langle 2\cos\theta_0\sin\varphi_0,\ 2\sin\theta_0\sin\varphi_0,\ 2\cos\varphi_0\rangle$$

$$2\cos\varphi_0=\sqrt{2}$$
$$\cos\varphi_0=\frac{\sqrt{2}}{2} \qquad \boxed{\varphi_0=\pi/4}$$

$$2\cos\theta_0\cancel{\sin\varphi_0}^{\frac{\sqrt{2}}{2}}=1$$
$$\cos\theta_0=\frac{1}{\sqrt{2}}, \qquad \boxed{\theta_0=\pi/4}$$

② Find the normal vector $\vec{N}\left(\frac{\pi}{4},\frac{\pi}{4}\right)=\vec{r}_\theta\left(\frac{\pi}{4},\frac{\pi}{4}\right)\times\vec{r}_\varphi\left(\frac{\pi}{4},\frac{\pi}{4}\right)$

$$\vec{r}(\theta,\varphi)=\langle 2\cos\theta\sin\varphi,\ 2\sin\theta\sin\varphi,\ 2\cos\varphi\rangle$$

$$\vec{r}_\theta=\langle -2\sin\theta\sin\varphi,\ 2\cos\theta\sin\varphi,\ 0\rangle$$

$$\vec{r}_\theta\left(\frac{\pi}{4},\frac{\pi}{4}\right)=\langle -1,1,0\rangle$$

$$\vec{r}_\varphi=\langle 2\cos\theta\cos\varphi,\ 2\sin\theta\cos\varphi,\ -2\sin\varphi\rangle$$

$$\vec{r}_\varphi\left(\frac{\pi}{4},\frac{\pi}{4}\right)=\langle 1,1,-\sqrt{2}\rangle$$

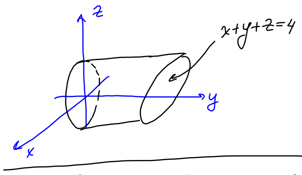
$$\vec{N}\left(\frac{\pi}{4}\times\frac{\pi}{4}\right)=\langle -1,1,0\rangle\times\langle 1,1,-\sqrt{2}\rangle=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ -1 & 1 & 0\\ 1 & 1 & -\sqrt{2}\end{vmatrix}$$

$$=-\sqrt{2}\,\vec{i}-\vec{j}\sqrt{2}-2\vec{k}=\langle -\sqrt{2},-\sqrt{2},-2\rangle$$

③ Equation of the tangent plane:

$$-\sqrt{2}(x-1)-\sqrt{2}(y-1)-2(z-\sqrt{2})=0$$

8. Find the area of the part of the cylinder $x^2+z^2=1$ which lies between the planes $y=0$ and $x+y+z=4$.



$$S.A.=\iint_S dS, \quad S:\vec{r}(u,v),\ (u,v)\in D$$

surface integral

$$=\iint_D |\vec{r}_u\times\vec{r}_v|\,dA$$

① Parametric equation for the cylinder: since $x^2+z^2=1$, then

$$\begin{cases} x=\cos\theta \\ y=y \\ z=\sin\theta \end{cases}$$

$$0\le y\le 4-x-z$$
$$0\le y\le 4-\sin\theta-\cos\theta$$
$$0\le\theta\le 2\pi$$

$$dA=dy\,d\theta$$

$$\vec{r}(y,\theta)=\langle\cos\theta,\ y,\ \sin\theta\rangle$$

② $$\vec{r}_y=\langle 0,1,0\rangle$$

$$\vec{r}_\theta=\langle -\sin\theta,\ 0,\ \cos\theta\rangle$$

$$\vec{r}_y\times\vec{r}_\theta=\langle 0,1,0\rangle\times\langle -\sin\theta,0,\cos\theta\rangle$$
$$=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ 0 & 1 & 0\\ -\sin\theta & 0 & \cos\theta\end{vmatrix}=\langle\cos\theta,\ 0,\ \sin\theta\rangle$$

$$|\vec{r}_y\times\vec{r}_\theta|=\sqrt{\cos^2\theta+\sin^2\theta}=1$$

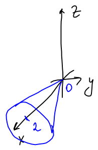
③ $$S.A.=\iint_D|\vec{r}_y\times\vec{r}_\theta|\,dA=\int_0^{2\pi}\int_0^{4-\sin\theta-\cos\theta}dy\,d\theta$$

$$=\int_0^{2\pi}(4-\sin\theta-\cos\theta)\,d\theta$$

$$=4\theta\Big]_0^{2\pi}+\cos\theta\Big]_0^{2\pi}-\sin\theta\Big]_0^{2\pi}$$

$$=\boxed{8\pi}$$

9. Find the area of the portion of the cone $x^2 = y^2 + z^2$ between the planes $x = 0$ and $x = 2$.



$x^2 = y^2 + z^2$

$x = \sqrt{y^2 + z^2}$

cylindrical coord.

$x = x$

$y = r\cos\theta$

$z = r\sin\theta$

since $x^2 = y^2 + z^2$

$x^2 = r^2$ or $x = r$

Vector equation of the cone is:

$\vec{R}(r,\theta) = \langle r, r\cos\theta, r\sin\theta \rangle$

$0 \le x \le 2$. since $x = r$, then $0 \le r \le 2$, $0 \le \theta \le 2\pi$ $= D$

$S.A. = \iint_D |\vec{R}_r \times \vec{R}_\theta| \, dA$

$\vec{R}_r = \langle 1, \cos\theta, \sin\theta \rangle$

$\vec{R}_\theta = \langle 0, -r\sin\theta, r\cos\theta \rangle$

$$\vec{R}_r \times \vec{R}_\theta = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & \cos\theta & \sin\theta \\ 0 & -r\sin\theta & r\cos\theta \end{vmatrix} = \vec{i}(r\cos^2\theta + r\sin^2\theta) - \vec{j}\, r\cos\theta - \vec{k}\, r\sin\theta$$

$= \langle r, -r\cos\theta, -r\sin\theta \rangle$

$|\vec{R}_r \times \vec{R}_\theta| = \sqrt{r^2 + r^2\cos^2\theta + r^2\sin^2\theta} = r\sqrt{2}$

$r^2 + r^2(\cos^2\theta + \sin^2\theta)$

$r^2 + r^2$

$S.A. = \int_0^{2\pi}\int_0^2 r\sqrt{2}\, r\,dr\,d\theta$

$= \sqrt{2}\int_0^{2\pi}\int_0^2 r^2\,dr\,d\theta$

$= \sqrt{2}\,(2\pi)\,\left.\frac{r^3}{3}\right]_0^2$

$= \boxed{\frac{16\sqrt{2}\,\pi}{3}}$