Chapter 11. **Three dimensional analytic geometry and vectors.**
Section 11.3 **The cross product.**

If we tighten a bolt by applying a force to a wrench, we produce a turning effect called a *torque* $\vec{\tau}$ that acts along the axis of the bolt to move it forward. The magnitude of the torque depends on two things:

- The distance from the axis of the bolt to the point where the force is applied. This is $|\vec{r}|$, the length of the position vector $\vec{r}$.
- The scalar component of the force $\vec{F}$ in the direction perpendicular to $\vec{r}$. This is the only component that can cause a rotation and it is

$$|\vec{F}| \sin\theta$$

  where θ is an angle between the vectors $\vec{r}$ and $\vec{F}$.

We define the magnitude of the torque to be the product of these two factors:

$$|\tau| = |\vec{r}||\vec{F}| \sin\theta$$

If $\vec{n}$ is a unit vector that points in the direction in which a right-threaded bolt moves, we define the torque to be the vector

$$\vec{\tau} = (|\vec{r}||\vec{F}| \sin\theta)\vec{n}.$$

We denote this torque vector by $\vec{\tau} = \vec{r} \times \vec{F}$ and we call it the *cross product* or *vector product* of $\vec{r}$ and $\vec{F}$.

Definition. If $\vec{a}$ and $\vec{b}$ are nonzero three-dimensional vectors, the **cross product** of $\vec{a}$ and $\vec{b}$ is the vector

$$\vec{a} \times \vec{b} = (|\vec{a}||\vec{b}| \sin\theta)\vec{n}$$

where θ is the angle between $\vec{a}$ and $\vec{b}$ and $\vec{n}$ is a unit vector perpendicular to both $\vec{a}$ and $\vec{b}$ and whose direction is given by the **right-hand rule:** If the fingers of your hand curl through the angle θ from $\vec{a}$ to $\vec{b}$, then your thumb points in the direction of $\vec{n}$.

If either $\vec{a}$ or $\vec{b}$ is $\vec{0}$, then we define $\vec{a} \times \vec{b}$ to be $\vec{0}$.

$\vec{a} \times \vec{b}$ is orthogonal to both $\vec{a}$ and $\vec{b}$.

Two nonzero vectors $\vec{a}$ and $\vec{b}$ are parallel if and only if $\vec{a} \times \vec{b} = \vec{0}$.

Properties of the cross product. If $\vec{a}$, $\vec{b}$, and $\vec{c}$ are vectors and k is a scalar, then

1. $\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$
2. $(k\vec{a}) \times \vec{b} = k(\vec{a} \times \vec{b}) = \vec{a} \times (k\vec{b})$
3. $\vec{a} \times (\vec{b} + \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$
4. $(\vec{a} + \vec{b}) \times \vec{c} = \vec{a} \times \vec{c} + \vec{b} \times \vec{c}$

The length of the cross product $\vec{a} \times \vec{b}$ is equal to the area of the parallelogram determined by $\vec{a}$ and $\vec{b}$.

The cross product in component form.

A **determinant of order 2** is defined by

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

A **determinant of order 3** can be defined in terms of second-order determinants as follows:

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

The cross product of a $\vec{a} = a_1\vec{\imath} + a_2\vec{\jmath} + a_3\vec{k}$ and $\vec{b} = b_1\vec{\imath} + b_2\vec{\jmath} + b_3\vec{k}$ is

$$\begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \vec{\imath}\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - \vec{\jmath}\begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + \vec{k}\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} =$$

$$(a_2b_3 - a_3b_2)\vec{\imath} - (a_1b_3 - a_3b_1)\vec{\jmath} + (a_1b_2 - a_2b_1)\vec{k}$$

Example 1. If $\vec{a} =< -2, 3, 4 >$ and $\vec{b} =< 3, 0, 1 >$, find $\vec{a} \times \vec{b}$.

Example 2. Find the area of the triangle with vertices $A(1, 2, 3)$, $B(2, -1, 1)$, $C(0, 1, -1)$.

Example 3. Find two unit vectors orthogonal to both $\vec{\imath} + \vec{\jmath}$ and $\vec{\imath} - \vec{\jmath} + \vec{k}$.

Triple products

The product $\vec{a} \cdot (\vec{b} \times \vec{c})$ is called the **scalar triple product** of the vectors $\vec{a}$, $\vec{b}$, and $\vec{c}$.

The volume of the parallelepiped determined by the vectors $\vec{a}$, $\vec{b}$, and $\vec{c}$ is the magnitude of their scalar triple product:

$$V = |\vec{a} \cdot (\vec{b} \times \vec{c})|.$$

$$\vec{a} \cdot (\vec{b} \times \vec{c}) = (\vec{a} \times \vec{b}) \cdot \vec{c}$$

Suppose that $\vec{a}$, $\vec{b}$, and $\vec{c}$ are given in component form:

$$\vec{a} =< a_1, a_2, a_3 >, \quad \vec{b} =< b_1, b_2, b_3 >, \quad \vec{c} =< c_1, c_2, c_3 > .$$

Then

$$\vec{a} \cdot (\vec{b} \times \vec{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Example 4. Find the volume of the parallelepiped determined by vectors $\vec{a} = 2\vec{\imath} + 3\vec{\jmath} - 2\vec{k}$, $\vec{b} = \vec{\imath} - \vec{\jmath}$, and $\vec{c} = 2\vec{\imath} + 3\vec{k}$.

Example 5. Use the scalar triple product to verify that the vectors $\vec{a} = 2\vec{\imath} + 3\vec{\jmath} + \vec{k}$, $\vec{b} = \vec{\imath} - \vec{\jmath}$, and $\vec{c} = 7\vec{\imath} + 3\vec{\imath} + 2\vec{k}$ are complanar; that is, they lie in the same plane.

The product $\vec{a} \times (\vec{b} \times \vec{c})$ is called the **vector triple product** of the vectors $\vec{a}$, $\vec{b}$, and $\vec{c}$.

$$\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c})\vec{b} - (\vec{a} \cdot \vec{b})\vec{c}.$$