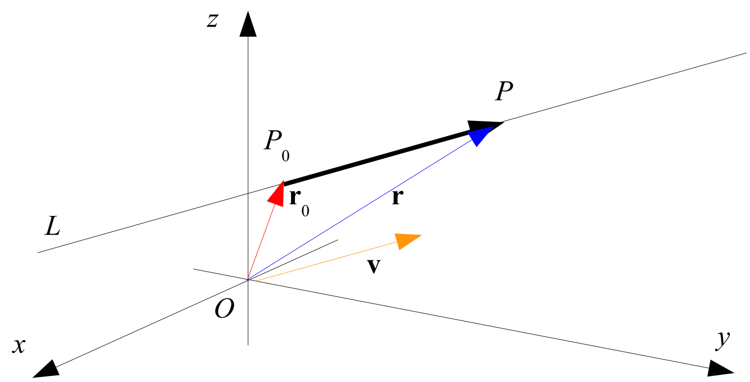
Chapter 11. **Three dimensional analytic geometry and vectors.**
Section 11.4 **Equations of lines and planes.**

A line L in 3D space is determined when we know a point $P_0(x_0, y_0, z_0)$ on L and the direction of L. Let $\vec{v}$ be a vector parallel to L, $P(x, y, z)$ be an arbitrary point on L and $\vec{r_0}$ and $\vec{r}$ be position vectors of P_0 and P.



$\vec{r} = \vec{r_0} + \vec{P_0P}$. Since $\vec{P_0P}$ is parallel to $\vec{v}$, there is a scalar t such that $\vec{P_0P} = t\vec{v}$. Thus a **vector equation** of L is

$$\vec{r} = \vec{r_0} + t\vec{v}.$$

Each value of the **parameter** t gives the position vector $\vec{r}$ of a point on L.

If $\vec{r_0} =< x_0, y_0, z_0 >$, $\vec{r} =< x, y, z >$, and $\vec{v} =< a, b, c >$, then

$$< x, y, z >=< x_0 + ta, y_0 + tb, z_0 + tc >$$

or

$$x = x_0 + ta, \quad y = y_0 + tb, \quad z = z_0 + tc$$

where $t \in \mathbb{R}$. These equations are called **parametric equations** of the line L through the point $P_0(x_0, y_0, z_0)$ and parallel to the vector $\vec{v} =< a, b, c >$.

If a vector $\vec{v} =< a, b, c >$ is used to describe the direction of a line L, then the numbers a, b, and c are called **direction numbers** of L.

Example 1. Find the vector equation and parametric equations for the line passing through the point $P(1, -1, -2)$ and parallel to the vector $\vec{v} = 3\vec{\imath} - 2\vec{\jmath} + \vec{k}$.

Since vectors $\vec{v} =< a, b, c >$ and $\vec{P_0P} =< x - x_0, y - y_0, z - z_0 >$ are parallel, then

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

These equations are called **symmetric equations** of L. If one of a, b, or c is 0, we can still write symmetric equations. For instance, if $c = 0$, then the symmetric equations of L are

$$\frac{x - x_0}{a} = \frac{y - y_0}{b}, \quad z = z_0.$$

This means that L lies in the plane $z = z_0$.

Example 2. Find symmetric equations for the line passing through the given points:

(a) $(2, -6, 1)$, $(1, 0, -2)$

(b) $(-1, 2, -4)$, $(-1, -3, 2)$

Example 3. Find symmetric equations for the line that passes through the point $(0, 2, -1)$ and is parallel to the line with parametric equations $x = 1 + 2t$, $y = 3t$, and $z = 5 - 7t$.

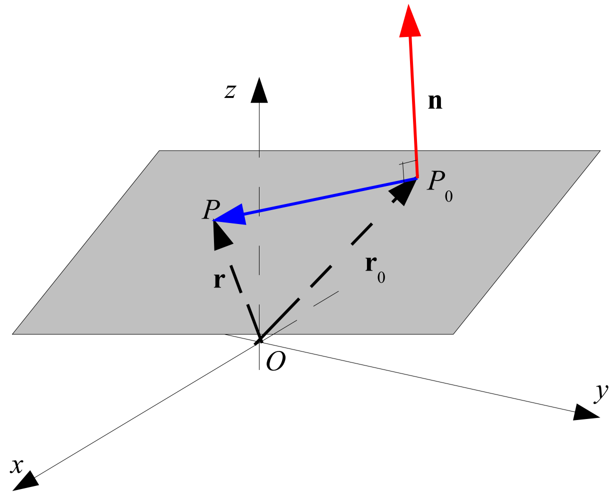
Example 4. Determine whether the lines L_1 and L_2 are parallel, skew (do not intersect and are not parallel), or intersecting. If they intersect, find the point of intersection.

(a) $L_1: \dfrac{x-4}{2} = \dfrac{y+5}{4} = \dfrac{z-1}{-3}$, $L_2: \dfrac{x-2}{1} = \dfrac{y+1}{3} = \dfrac{z}{3}$

(b) $L_1: \dfrac{x-1}{2} = \dfrac{y}{1} = \dfrac{z-1}{4}$, $L_2: \dfrac{x}{1} = \dfrac{y+2}{2} = \dfrac{z+2}{3}$

(c) $L_1: \ x = -6t, y = 1 + 9t, z = -3t,$
$L_2: \ x = 1 + 2s, y = 4 - 3s, z = s.$

A plane in space is determined by a point $P_0(x_0, y_0, z_0)$ in the plane and a vector $\vec{n}$ that is orthogonal to the plane. n is called a **normal vector**. Let $P(x, y, z)$ be an arbitrary point in the plane and $\vec{r}$ and $\vec{r_0}$ be the position vectors of P and P_0.



$\vec{P_0P} = \vec{r} - \vec{r_0}$. The normal vector $\vec{n}$ is orthogonal to every vector in the given plane, in particular, $\vec{n}$ is orthogonal to $\vec{P_0P}$.

$$\vec{n} \cdot (\vec{r} - \vec{r_0}) = 0 \quad \text{or} \quad \vec{n} \cdot \vec{r} = \vec{n} \cdot \vec{r_0}$$

Either of the two equations is called a **vector equation of the plane**.

Let $\vec{n} =< a, b, c >$, $\vec{r_0} =< x_0, y_0, z_0 >$, and $\vec{r} =< x, y, z >$, then

$$\vec{v} \cdot \vec{r} - \vec{r_0} = a(x - x_0) + b(y - y_0) + c(z - z_0)$$

so we can rewrite the vector equation in the following way:

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

This equation is called the **scalar equation of the plane through** $P_0(x_0, y_0, z_0)$ **with normal vector** $\vec{n} =< a, b, c >$.

By collecting terms in a scalar equation of a plane, we can rewrite the equation as

$$ax + by + cz + d = 0,$$

where $d = -ax_0 - by_0 - cz_0$.

Example 5. Find the equation of the plane through the point $P_0(-5, 1, 2)$ with the normal vector $\vec{n} =< 3, -3, -1 >$.

Example 6. Find the equation of the plane passing through the points $(-1, 1, -1)$, $(1, -1, 2)$, $(4, 0, 3)$.

Example 7. Find an equation of the plane that passes through the point $(1, 6, -4)$ and contains the line $x = 1 + 2t$, $y = 2 - 3t$, $z = 3 - t$.

Two planes are **parallel** if their normal vectors are parallel. If two planes are not parallel, then they intersect in a straight lineand the angle between the two planes is defined as the acute angle between their normal vectors.

Example 8. (a) Find the angle between the planes $x - 2y + z = 1$ and $2x + y + z = 1$.

(b)Find symmetric equation for the line of intersection of the planes.

Problem. Find a formula for the distance D from a point $P_1(x_1, y_1, z_1)$ to the plane $ax + by + cz + d = 0$.

$$D = \frac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{a^2 + b^2 + c^2}}$$

Example 9. Find the distance between the parallel planes $x+2y-z = -1$ and $3x+6y-3z = 4$.