Chapter 11. **Three dimensional analytic geometry and vectors.**
Section 11.5 **Quadric surfaces.**

Curves in $\mathbb{R}^2$:
ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

parabola $y = ax^2$ or $x = by^2$

A **quadric surface** is the graph of a second degree equation in three variables. The most general such equation is

$$Ax^2 + By^2 + Cz^2 + Dxy + Exz + Fyz + Gx + Hy + Iz + J = 0,$$

where $A, B, C, ..., J$ are constants. By translation and rotation the equation can be brought into one of two standard forms

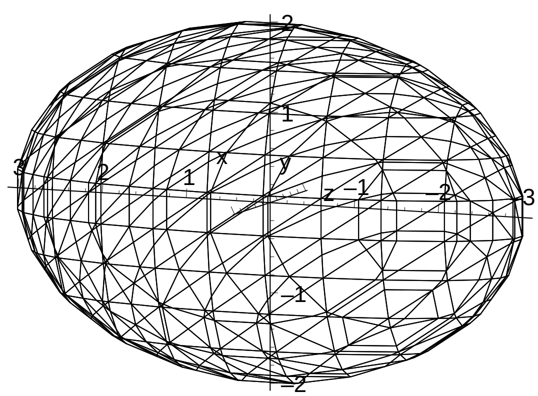
$$Ax^2 + By^2 + Cz^2 + J = 0 \quad \text{or} \quad Ax^2 + By^2 + Iz = 0$$

In order to sketch the graph of a quadric surface, it is useful to determine the curves of intersection of the surface with planes parallel to the coordinate planes. These curves are called **traces** of the surface.

Ellipsoids The quadric surface with equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

is called an ellipsoid because all of its traces are ellipses.



The six intercepts of the ellipsoid are $(\pm a, 0, 0)$, $(0, \pm b, 0)$, and $(0, 0, \pm c)$ and the ellipsoid lies in the box

$$|x| \leq a, \quad |y| \leq b, \quad |z| \leq c$$

Since the ellipsoid involves only even powers of x, y, and z, the ellipsoid is symmetric with respect to each coordinate plane.

Example 1. Find the traces of the surface

$$4x^2 + 9y^2 + 36z^2 = 36$$

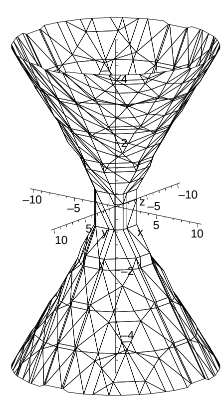
in the planes $x = k$, $y = k$, and $z = k$. Identify the surface and sketch it.

Hyperboloids

Hyperboloid of one sheet. The quadric surface with equations

1. $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} - \dfrac{z^2}{c^2} = 1$
2. $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$
3. $-\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$

are called **hyperbolids of one sheet** since two of its traces are hyperbolas and one is an ellipse, and its surface consists of just one piece.



Example 2. Find the traces of the surface

$$x^2 - y^2 + z^2 = 1$$

in the planes $x = k$, $y = k$, and $z = k$. Identify the surface and sketch it.

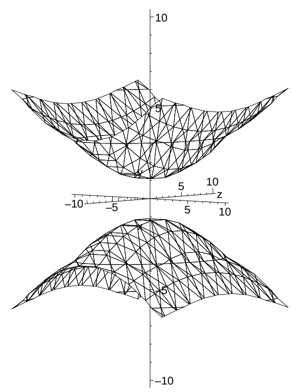
Hyperboloid of two sheets. The quadric surface with equations

1. $-\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$

2. $-\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} - \dfrac{z^2}{c^2} = 1$

3. $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} - \dfrac{z^2}{c^2} = 1$

are called **hyperbolids of two sheets** since two of its traces are hyperbolas and one is an ellipse, and its surface is two sheets.



Example 3. Find the traces of the surface

$$9x^2 - y^2 - z^2 = 9$$

in the planes $x = k$, $y = k$, and $z = k$. Identify the surface and sketch it.

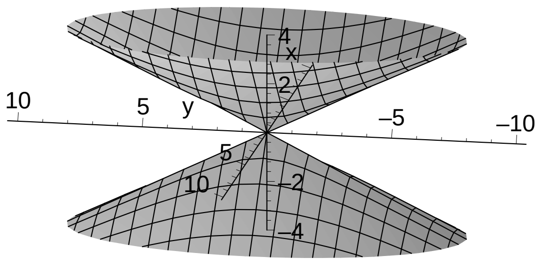
Cone. The quadric surface with equations

1. $\dfrac{z^2}{c^2} = \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2}$

2. $\dfrac{y^2}{b^2} = \dfrac{x^2}{a^2} + \dfrac{z^2}{c^2}$

3. $\dfrac{x^2}{a^2} = \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2}$

are called **cones**.



If P is any point on the cone, then the line OP lies entirely on the cone. The traces in horizontal planes $z = k$ are ellipses and traces in vertical planes $x = k$ or $y = k$ are hyperbolas if $k \neq 0$ but are pairs of lines if $k = 0$.

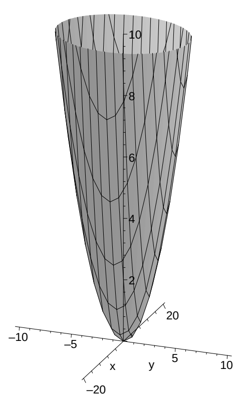
Paraboloids. The quadric surface with equations

1. $\dfrac{z}{c} = \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2}$

2. $\dfrac{y}{b} = \dfrac{x^2}{a^2} + \dfrac{z^2}{c^2}$

3. $\dfrac{x}{a} = \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2}$

are called **elliptic paraboloids** because its traces in horizontal planes $z = k$ are ellipses, whereas its in vertical planes $x = k$ or $y = k$ are parabolas.



Example 4. Find the traces of the surface

$$y = x^2 + z^2$$

in the planes $x = k$, $y = k$, and $z = k$. Identify the surface and sketch it.

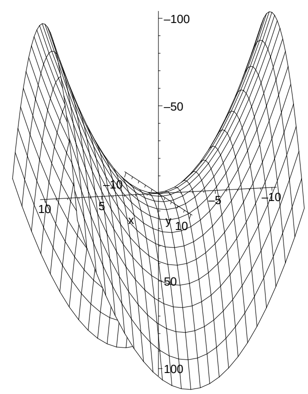
The **hyperbolic paraboloid**

1. $\dfrac{z}{c} = \dfrac{x^2}{a^2} - \dfrac{y^2}{b^2}$

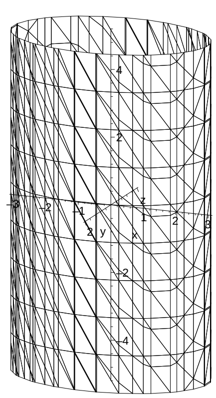
2. $\dfrac{y}{b} = \dfrac{x^2}{a^2} - \dfrac{z^2}{c^2}$

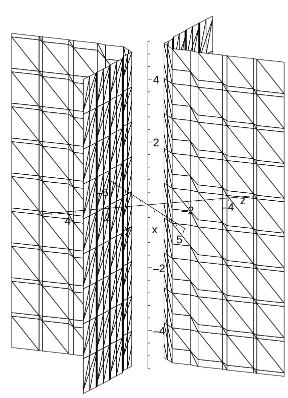
3. $\dfrac{x}{a} = \dfrac{y^2}{b^2} - \dfrac{z^2}{c^2}$

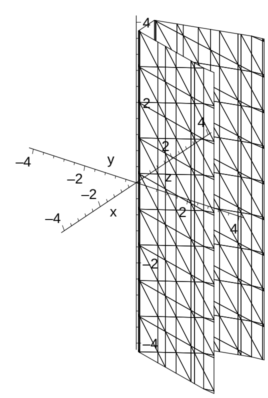
also has parabolas as its vertical traces, but its has hyperbolas as its horizontal traces.



Quadric cylinders. When one of the variables is missing from the equation of a surface, then the surface is a cylinder. The equation $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ represents the elliptic cylinder, the equation $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ represents a hyperbolic cylinder, and the equation $y = ax^2$ represents the hyperbolic cylinder.







Example 5. Classify the quadric surface $4x^2 - y^2 + z^2 + 8x + 8z + 24 = 0$ and sketch it.