

Chapter 11. **Three dimensional analytic geometry and vectors.**

Section 11.7 **Arc length and curvature.**

The length of a space curve with the vector equation $\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$, $a \leq t \leq b$, if the curve is traversed exactly once as t increases from a to b , is

$$L = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2 + [h'(t)]^2} dt = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} dt = \int_a^b |\vec{r}'(t)| dt$$

Example 1. Find the length of the curve given by the vector function $\vec{r}(t) = \langle e^t, e^t \sin t, e^t \cos t \rangle$, $0 \leq t \leq 2\pi$.

A curve given by a vector function $\vec{r}(t)$ on an interval I is called **smooth** if \vec{r}' is continuous and $\vec{r}'(t) \neq \vec{0}$ (except possibly at any endpoints of I). A curve that is made up of a finite number of smooth pieces is called **piecewise smooth**. The arc length formula holds for piecewise-smooth functions.

A single curve C can be represented by more than one vector function. Arc length is independent of the parametrization that is used.

Suppose that C is a piecewise-smooth curve given by a vector function $\vec{r}(t) = f(t)\vec{i} + g(t)\vec{j} + h(t)\vec{k}$, $a \leq t \leq b$ and at least one of f , g , h is one-to-one on (a, b) . We define its **arc length function** s by

$$s(t) = \int_a^t |\vec{r}'(t)| dt = \int_a^t \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} dt$$

It is often useful to **parametrize a curve with respect to arc length**.

Example 2. Reparametrize the curve $\vec{r}(t) = \langle 1 + 2t, 3 + t, -5t \rangle$ with respect to arc length measured from the point where $t = 0$ in the direction of increasing t .

Curvature. If C is a smooth curve defined by the vector function \vec{r} , then $\vec{r}'(t) \neq \vec{0}$. The unit tangent vector

$$\vec{T}(t) = \frac{\vec{r}'(t)}{|\vec{r}'(t)|}$$

indicates the direction of the curve.

The curvature of C at a given point is a measure of how quickly the curve changes direction at that point. We define it to be the magnitude of the rate of change of the unit tangent vector with respect to arc length.

Definition. The **curvature** of a curve is

$$\kappa = \left| \frac{d\vec{T}(t)}{ds} \right|$$

where \vec{T} is the unit tangent vector.

Since $\frac{ds}{dt} = |\vec{r}'(t)|$, then

$$\kappa(t) = \frac{|\vec{T}'(t)|}{|\vec{r}'(t)|}$$

Example 3. Find the curvature of a circle of radius a .

Theorem. The curvature of the curve given by the vector function \vec{r} is

$$\kappa(t) = \frac{|\vec{r}'(t) \times \vec{r}''(t)|}{|\vec{r}'(t)|^3}$$

Example 4. Find the curvature of the curve given by the vector function

$$\vec{r}(t) = (t^2 + 2)\vec{i} + (t^2 - 4t)\vec{j} + 2t\vec{k}.$$

For the special case of a plane curve with equation $y = f(x)$, the curvature

$$\kappa(x) = \frac{|f''(x)|}{[1 + (f'(x))^2]^{3/2}}$$

Example 5. Find the curvature of the function $y = \sqrt{x}$.

The normal and binormal vectors. At a given point on a smooth space curve $\vec{r}(t)$, there are many vectors that are orthogonal to the unit tangent vector $\vec{T}(t)$. Since $|\vec{T}(t)| = 1$, then $\vec{T}'(t)$ is orthogonal to $\vec{T}(t)$. We can define the **principal unit normal vector** $\vec{N}(t)$ as

$$\vec{N}(t) = \frac{\vec{T}'(t)}{|\vec{T}'(t)|}$$

The vector $\vec{B}(t) = \vec{T}(t) \times \vec{N}(t)$ is called the **binormal vector**. $|\vec{B}(t)| = 1$.

Example 6. Find the unit normal and binormal vectors for the curve given by $\vec{r}(t) = \langle t^2, 2t, \ln t \rangle$.

The plane determined by the normal and binormal vectors \vec{N} and \vec{B} at a point P on a curve C is called the **normal plane** of C at P . It consists of all lines that are orthogonal to the tangent vector \vec{T} . The plane determined by vectors \vec{T} and \vec{N} is called the **osculating plane** of C at P . It is the plane that comes closest to containing the part of the curve near P .

The circle that lies in the osculating plane of C at P , has the same tangent as C at P , lies on the concave side of C (toward which \vec{N} points), and has radius $\rho = 1/\kappa$ is called the **osculating circle** (or **the circle of curvature**) of C at P .

Example 7. Find the equations of the normal plane and osculating plane of the curve given by $x = t$, $y = t^2$, and $z = t^3$ at the point $P(1, 1, 1)$.