Chapter 12. **Partial derivatives.**
Section 12.3 **Partial derivatives.**

Definition. If f is a function of two variables, its **partial derivatives** are the functions f_x and f_y defined by

$$f_x(x,y) = \lim_{h\to 0} \frac{f(x+h,y) - f(x,y)}{h}$$

$$f_y(x,y) = \lim_{h\to 0} \frac{f(x,y+h) - f(x,y)}{h}$$

Notation for partial derivatives: If $z = f(x,y)$, we write

$$f_x(x,y) = f_x = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x} f(x,y) = \frac{\partial z}{\partial x}$$

$$f_y(x,y) = f_y = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y} f(x,y) = \frac{\partial z}{\partial y}$$

Rule for finding partial derivatives of $z = f(x,y)$:

1. To find f_x, regards y as a constant and differentiate $f(x,y)$ with respect to x.
2. To find f_y, regards x as a constant and differentiate $f(x,y)$ with respect to y.

Example 1. Find the first partial derivatives of the following functions:
(a) $f(x,y) = x^4 + x^2y^2 + y^4$

(b) $f(x,y) = x^y$

(c) $f(x,y) = e^x \tan(x-y)$

Example 2. Find $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$ if z is define implicitly as a function of x and y by the equation

$$xyz = \cos(x + y + z)$$

Functions of more that two variables. Partial derivatives can also be defined for functions of three or more variables.

If f is a function of three variables x, y, and z, ten its partial derivative with respect to x can be defined as

$$\frac{\partial f}{\partial x} = f_x(x, y, z) = \lim_{h \to 0} \frac{f(x + h, y, z) - f(x, y, z)}{h}$$

and it is found by regarding y and z as constants and differentiating $f(x, y, z)$ with respect to x.

In general, if u is function of n variables, $u = f(x_1, x_2, ..., x_n)$, its partial derivative with respect to x_i is

$$\frac{\partial u}{\partial x_i} = f_{x_i}(x_1, x_2, ..., x_n) = \lim_{h \to 0} \frac{f(x_1, ..., x_i + h, , , x_n) - f(x_1, ..., x_i, , , x_n)}{h}$$

Higher derivatives. If $z = f(x, y)$, then its **second partial derivatives** are defined as

$$(f_x)_x = f_{xx} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 z}{\partial x^2}$$

$$(f_x)_y = f_{xy} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 z}{\partial x \partial y}$$

$$(f_y)_x = f_{yx} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 z}{\partial y \partial x}$$

$$(f_y)_y = f_{yy} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2} = \frac{\partial^2 z}{\partial y^2}$$

Clairaut's Theorem Suppose f is defined on a disk D that contains the point (a, b). If the functions f_{xy} and f_{yx} are both continuous on D, then

$$f_{xy}(a, b) = f_{yx}(a, b).$$

Example 3. Find all the second partial derivatives for the function $f(x, y) = (x^2 + y^2)^{3/2}$

Example 4. Determine whether the function $u = e^{-x} \cos y - e^{-y} \cos x$ is a solution of Laplace's equation $u_{xx} + u_{yy} = 0$

Example 5. Find f_{xyz} for the function $f(x, y, z) = e^{xyz}$.