Chapter 12. **Partial derivatives.**
Section 12.4 **Tangent planes and differentials.**

Tangent planes.

Suppose a surface S has equation $z = f(x, y)$, where f has continuous first partial derivatives, and let $P(x_0, y_0, z_0)$ be a point on S. Let C_1 and C_2 be the curves obtained by intersecting the vertical planes $y = y_0$ and $x = x_0$ with the surface S. P lies on both C_1 and C_2. Let T_1 and T_2 be the tangent lines to the curves C_1 and C_2 Let T_1 and T_2 be the tangent lines to the curves C_1 and C_2 at the point P. The **tangent plane** to the surface S at the point P is defined to be the plane that containts both of the tangent lines T_1 and T_2.

An **equation on the tangent plane to the surface** $z = f(x, y)$ **at the point** $P(x_0, y_0, z_0)$ is

$$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

Example 1. Find the equation of the tangent plane to the surface $z = \ln(2x + y)$ at the point $(-1, 3, 0)$.

Differentials. Consider a function of two variables $z = f(x, y)$. If x and y are given increments Δx and Δy, then the corresponding **increment** of z is

$$\Delta z = f(x + \Delta x, y + \Delta y) - f(x, y)$$

The increment Δz represents the change in the value of f when (x, y) changes to $(x + \Delta x, y + \Delta y)$.

The **differentials** dx and dy are independent variables. The **differential** dz (or the **total differential**), is defined by

$$dz = f_x(x, y)dx + f_y(x, y)dy$$

Example 2. Find the differential of the function $z = \sqrt[3]{x + y^2}$.

If we take

$$dx = \Delta x = x - a \qquad dy = \Delta y = y - b$$

then the differential of z is

$$dz = f_x(a,b)(x-a) + f_y(a,b)(y-b)$$

On the other hand, the equation of the tangent plane to the surface $z = f(x,y)$ at the point $(a, b, f(a,b))$ is

$$z - f(a,b) = f_x(a,b)(x-a) + f_y(a,b)(y-b)$$

We see that dz represents the change of height of the tangent plane whereas Δz represent the change in height of the surface $z = f(x,y)$ when (x,y) changes from (a,b) to $(a+\Delta x, b+\Delta y)$.

If $dx = \Delta x$ and $dy = \Delta y$ are small, then $\Delta z \approx dz$ and

$$f(a+\Delta x, b+\Delta y) \approx f(a,b) + dz$$

Example 3.

1. Use differential to approximate the value of the function $f(x,y) = \sqrt{20 - x^2 - 7y^2}$ at the point $(1.95, 1.08)$.

2. Use differential to approximate the number $(\sqrt{99} + \sqrt[3]{124})^2$

Example 4. Use differentials to estimate the amount of tin in a closed tin can with diameter 8 cm and height 12 cm if the tin is 0.04 cm thick.

Functions of three or more variables. If $u = f(x, y, z)$, then the **increment** of u is

$$\Delta u = f_x(x + \Delta x, y + \Delta y, z + \Delta z) - f(x, y, z)$$

The **differential** du is defined in terms of the differentials dx, dy, and dz of the independent variables by

$$du = f_x(x, y, z)dx + f_y(x, y, z)dy + f_z(x, y, z)dz$$

If $dx = \Delta x$, $dy = \Delta y$, and $dz = \Delta z$ are small and f has continuous partial derivatives, then $\Delta u \approx du$.