Chapter 12. **Partial derivatives.**

Section 12.5 **The chain rule.**

For a function of a single variable $y = f(x)$ and $x = g(t)$, then $\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt}$.

The Chain Rule (case 1). Suppose that $z = f(x, y)$ is a differentiable function of x and y, where $x = g(t)$ and $y = h(t)$ are both differentiable functions of t. Then z is a differentiable function of t and

$$\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

Example 1. If $z = x^2y^3 + x^3y^2$, where $x = 1 + \sqrt{t}$ and $y = 1 + e^{2t}$, find $\frac{dz}{dt}$.

Example 2. The radius of a right circular cylinder is decreasing at a rate of 1.2 cm/s while its height is increasing at a rate of 3 cm/s. At what rate is the volume of the cylinder changing when the radius is 80 cm and the height is 150 cm.

The Chain Rule (case 2). Suppose that $z = f(x, y)$ is a differentiable function of x and y, where $x = g(s, t)$, $y = h(s, t)$, and the partial derivatives g_s, g_t, h_s, and h_t exist. Then

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}$$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$

Example 3. If $z = x^2 \sin y$, where $x = s^2 + t^2$ and $y = 2st$, find $\dfrac{\partial z}{\partial s}$ and $\dfrac{\partial z}{\partial t}$.

The Chain Rule (general version). Suppose that u is a differentiable function of the n variables $x_1, x_2, ..., x_n$ and each x_j is a function of the m variables $t_1, t_2, ..., t_m$ such that all partial derivatives $\partial x_j / \partial t_i$ exist ($j = 1, 2, ..., n$, $i = 1, 2, ..., m$). Then u is a function of $t_1, t_2, ..., t_m$ and

$$\frac{\partial u}{\partial t_i} = \frac{\partial u}{\partial x_1}\frac{\partial x_1}{\partial t_i} + \frac{\partial u}{\partial x_2}\frac{\partial x_2}{\partial t_i} + ... + \frac{\partial u}{\partial x_n}\frac{\partial x_n}{\partial t_i}$$

for each $i = 1, 2, ..., m$.

Example 4. If $z = \dfrac{x}{y}$, where $x = re^{st}$, $y = rse^t$, find $\dfrac{\partial z}{\partial r}$, $\dfrac{\partial z}{\partial s}$, and $\dfrac{\partial z}{\partial t}$.

Implicit differentiation. We suppose that an equation of the form $F(x, y) = 0$ defines y implicitly as a differentiable function of x, that is $y = f(x)$, where $F(x, f(x)) = 0$ for all x in the domain of f.

In order to find dy/dx, we differentiate both parts of the equation $F(x, y) = 0$ with respect to x:

$$\frac{d}{dx}(F(x, y)) = \frac{d}{dx}(0)$$

$$\frac{\partial F}{\partial x}\frac{dx}{dx} + \frac{\partial F}{\partial y}\frac{dy}{dx} = 0$$

Since $\frac{dx}{dx} = 1$, then

$$\frac{dy}{dx} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}}$$

Example 5. Find dy/dx if $x^2 - xy + y^3 = 8$.

Suppose that z is given implicitly as a function $z = f(x, y)$ by an equation of the form $F(x, y, z) = 0$. If F is differentiable and f_x and f_y exist, then we can use the Chain Rule to differentiate the equation $F(x, y, z) = 0$ as follows

$$\frac{\partial F}{\partial x}\frac{\partial x}{\partial x} + \frac{\partial F}{\partial y}\frac{\partial y}{\partial x} + \frac{\partial F}{\partial x}\frac{\partial x}{\partial x} = 0$$

Since $\frac{\partial x}{\partial x} = 1$ and $\frac{\partial y}{\partial x} = 0$, then

$$\frac{\partial F}{\partial x} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial x} = 0$$

or

$$\frac{\partial z}{\partial x} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial z}} \qquad \frac{\partial z}{\partial y} = -\frac{\frac{\partial F}{\partial y}}{\frac{\partial F}{\partial z}}$$

Example 6. Find $\partial z/\partial x$ and $\partial z/\partial y$ if $xyz = \cos(x + y + z)$.